Homology and cohomology **Spring 2009**
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Series 1

The purpose of this series is to study $\otimes$ and Hom. An introductory reference for tensor products is [1, Chapter V]. Let G and A be abelian groups. Their *tensor product* $G\otimes A$ is the free abelian group generated by the symbols $g\times a$ for $g\in G$ and $a\in A$ modulo the relations generated by

$$(1)\qquad (g+g')\times a\sim(g\times a)+(g'\times a),\quad g\times(a+a')\sim(g\times a)+(g\times a')$$

$$(2)\qquad gr\times a\sim g\times ra,\quad a,a'\in A,\quad g,g'\in G,\quad r\in\mathbb{Z}$$

Remark 1. We note that although relation (2) is implied by (1), it is conceptually useful to have (2) as part of the definition (for generalizing later to tensor products over arbitrary rings R).

In particular, $G\otimes A$ is an abelian group. The equivalence class of $g\times a$ in $G\otimes A$ is denoted by $g\otimes a$. By construction, every element of $G\otimes A$ can be written as a finite sum $\sum g_i\otimes a_i$, for $g_i\in G$ and $a_i\in A$. By construction, the elements $g\otimes a\in G\otimes A$ satisfy the following relations,

$$(3)\qquad (g+g')\otimes a=g\otimes a+g'\otimes a,\quad g\otimes(a+a')=g\otimes a+g\otimes a'$$

$$(4)\qquad gr\otimes a=g\otimes ra,\quad a,a'\in A,\quad g,g'\in G,\quad r\in\mathbb{Z}.$$

Sometimes $G\otimes A$ is denoted by $G\otimes_{\mathbb{Z}}A$; the notation emphasizes relation (4).

Exercise 1. Let A be an abelian group. Use relations (3)-(4) to prove the following.

(a) $0\otimes a=0=a\otimes 0$, for every $a\in A$.
(b) Conclude that $0\otimes A=0=A\otimes 0$.

Exercise 2. Define $\mathbb{Z}_n:=\mathbb{Z}/n\mathbb{Z}$ for each $n\geq 1$. Use relations (3)-(4) to prove the following.

(a) $\mathbb{Z}_2\otimes\mathbb{Q}=0$.
(b) $\mathbb{Z}_n\otimes\mathbb{Q}=0$ for every $n\geq 1$.
(c) $M\otimes\mathbb{Q}=0$ for every finite abelian group M.
(d) $\mathbb{Z}_2\otimes\mathbb{Z}_3=0$.

If M is an abelian group, then a map of sets $f:G\times A\longrightarrow M$ is called $\mathbb{Z}$-*bilinear* (or bilinear) if the following are satisfied:

$$(5)\qquad f(g+g',a)=f(g,a)+f(g',a),\quad f(g,a+a')=f(g,a)+f(g,a'),$$

$$(6)\qquad f(gr,a)=f(g,ra),\quad a,a'\in A,\quad g,g'\in G,\quad r\in\mathbb{Z}.$$

Remark 2. We note that although property (6) is implied by (5), it is conceptually useful to have (6) as part of the definition (for generalizing later to R-bilinear maps with R an arbitrary ring).

There is a naturally occurring bilinear map defined by

$$j : G \times A \longrightarrow G \otimes A, \quad (g, a) = g \times a \longmapsto g \otimes a$$

which satisfies the universal property: given any abelian group M and bilinear map f, there exists a unique homomorphism $\overline{f}$ of abelian groups which makes the diagram

$$(7) \qquad \begin{array}{ccc} G \times A & \xrightarrow{f} & M \\ {\scriptstyle j}\downarrow & \nearrow {\scriptstyle \exists! \, \overline{f}} & \\ G \otimes A & & \end{array}$$

commute.

Exercise 3. Prove that j satisfies the universal property (7).

The universal property of $G \otimes M$ characterizes this abelian group and the bilinear map $j : G \times A \longrightarrow G \otimes A$ uniquely (up to an isomorphism of $G \otimes A$). The universal property also makes it easy to construct homomorphisms out of $G \otimes A$.

Exercise 4. Let $\gamma : G \longrightarrow G'$ and $\alpha : A \longrightarrow A'$ be homomorphisms of abelian groups. Prove the following.

(a) The map $\gamma \otimes \alpha : G \otimes A \longrightarrow G' \otimes A'$ defined by $(\gamma \otimes \alpha)(g \otimes a) = \gamma g \otimes \alpha a$ is a well-defined homomorphism of abelian groups.
(b) $\mathrm{id}_G \otimes \mathrm{id}_A = \mathrm{id}_{G \otimes A}$.
(c) For composable homomorphisms, $\gamma' \gamma \otimes \alpha' \alpha = (\gamma' \otimes \alpha')(\gamma \otimes \alpha)$.
(d) $(\gamma_1 + \gamma_2) \otimes \alpha = \gamma_1 \otimes \alpha + \gamma_2 \otimes \alpha$.
(e) $\gamma \otimes (\alpha_1 + \alpha_2) = \gamma \otimes \alpha_1 + \gamma \otimes \alpha_2$.

This exercise proves that tensor product $- \otimes -$ is an additive functor.

Proposition 3. *If G is an abelian group and $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is an exact sequence of abelian groups, then*

$$G \otimes A \xrightarrow{\mathrm{id} \otimes \alpha} G \otimes B \xrightarrow{\mathrm{id} \otimes \beta} G \otimes C \to 0$$

is an exact sequence (of abelian groups).

Any exact sequence of the form $A \to B \to C \to 0$ is called a *right exact sequence*. Proposition 3 asserts that $G \otimes -$ preserves right exact sequences.

Exercise 5. Verify the details of the proof of Proposition 3 below.

Proof of Proposition 3. The idea is to compare $G \otimes C$ with the cokernel of $\mathrm{id} \otimes \alpha$ as follows.

$$\begin{array}{ccccccc} G \otimes A & \xrightarrow{\mathrm{id} \otimes \alpha} & G \otimes B & \xrightarrow{p} & \operatorname{coker}(\mathrm{id} \otimes \alpha) & \longrightarrow & 0 \\ \| & & \| & & \downarrow {\scriptstyle q} & & \| \\ G \otimes A & \xrightarrow[\mathrm{id} \otimes \alpha]{} & G \otimes B & \xrightarrow[\mathrm{id} \otimes \beta]{} & G \otimes C & \longrightarrow & 0 \end{array}$$

Since the composition $(\mathrm{id}\otimes\beta)(\mathrm{id}\otimes\alpha) = 0$, there exists a unique homomorphism q which makes the diagram commute. In particular, $qp(g\otimes b) = g\otimes\beta(b)$ for each $g \in G$ and $b \in B$. We want to show that q is an isomorphism; the idea is to look for a two-sided inverse to q. To construct a homomorphism $r : G\otimes C \longrightarrow \mathrm{coker}(\mathrm{id}\otimes\alpha)$, by the universal property it suffices to construct a bilinear map $r' : G \times C \longrightarrow \mathrm{coker}(\mathrm{id}\otimes\alpha)$. Since $\beta(B) = C$, for each $c \in C$ there exists $b \in B$ such that $\beta(b) = c$; by exactness at B, the definition $r'(g, c) := p(g\otimes b)$ gives a well-defined function. It is easy to check that r' is bilinear and that $qr = \mathrm{id}$ and $rq = \mathrm{id}$, hence q is an isomorphism. Since the top row is exact and is isomorphic to the bottom row, it follows that the bottom row is exact. □

Exercise 6. Let A be an abelian group. Define $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ for each $n \geq 1$. Prove the following.

(a) $A\otimes\mathbb{Z} \cong A$ and $\mathbb{Z}\otimes A \cong A$.
(b) $A\otimes\mathbb{Z}_n \cong A/nA$ for each $n \geq 1$.
(c) $\mathbb{Z}_m\otimes\mathbb{Z}_n \cong \mathbb{Z}_d$, with $d := \gcd(m, n)$.

To get started on (b), consider the exact sequence $\mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$. By Proposition 3, applying $A\otimes-$ gives an exact sequence

$$A\otimes\mathbb{Z} \to A\otimes\mathbb{Z} \to A\otimes\mathbb{Z}_n \to 0.$$

Exercise 7. Let A, B, G be abelian groups. Let $\{A_t\}_t$ be a collection of abelian groups indexed on a set. Prove the following.

(a) $G\otimes A \cong A\otimes G$.
(b) $(A\otimes B)\otimes G \cong A\otimes(B\otimes G)$.
(c) $G\otimes(A \oplus B) \cong (G\otimes A) \oplus (G\otimes B)$.
(d) $G\otimes\big(\oplus_t A_t\big) \cong \oplus_t (G\otimes A_t)$.
(e) $\mathbb{Z}^m\otimes\mathbb{Z}^n \cong Z^{mn}$, for any $m, n \geq 1$.

The universal property can be used to prove (a)-(d).

By Proposition 3, $G\otimes-$ preserves surjective maps. In general, $G\otimes-$ does not preserve injective maps.

Exercise 8. Define $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ for each $n \geq 1$, and consider the natural inclusion $\mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q}$ of abelian groups.

(a) Apply $\mathbb{Z}_n\otimes-$ and prove that the resulting map $\mathbb{Z}_n\otimes\mathbb{Z} \to \mathbb{Z}_n\otimes\mathbb{Q}$ is not injective for every $n \geq 2$.

A introductory reference on the Hom functor is [1, Chapter I]. Let A and B be abelian groups. The set

$$\mathrm{Hom}(A, B) := \{f \mid f : A \longrightarrow B\}$$

of all homomorphisms of abelian groups from A to B is an abelian group, with addition defined for $f, g : A \longrightarrow B$ by $(f + g)a = fa + ga$.

Exercise 9. Let A, B be abelian groups. Prove the following.

(a) $\operatorname{Hom}(A,0)=0$ and $\operatorname{Hom}(0,B)=0$.
(b) $\operatorname{Hom}(\mathbb{Z},B)\cong B$.

Exercise 10. Let $\alpha : A\longrightarrow A'$ and $\beta : B\longrightarrow B'$ be homomorphisms of abelian groups. Prove the following.

(a) The map $\beta_* : \operatorname{Hom}(A,B)\longrightarrow\operatorname{Hom}(A,B')$ defined by $\beta_*(f)=\beta f$

$$\big(i.e., \quad A\xrightarrow{f}B \quad\longmapsto\quad A\xrightarrow{f}B\xrightarrow{\beta}B'\big)$$

is a well-defined homomorphism of abelian groups.
(b) $(\mathrm{id}_B)_*=\mathrm{id}_{\operatorname{Hom}(A,B)}$
(c) For composable homomorphisms, $(\beta'\beta)_*=(\beta')_*(\beta)_*$.
(d) The map $\alpha^* : \operatorname{Hom}(A',B)\longrightarrow\operatorname{Hom}(A,B)$ defined by $\alpha^*(f')=f'\alpha$

$$\big(i.e., \quad A'\xrightarrow{f'}B \quad\longmapsto\quad A\xrightarrow{\alpha}A'\xrightarrow{f'}B\big)$$

is a well-defined homomorphism of abelian groups.
(e) $(\mathrm{id}_A)^*=\mathrm{id}_{\operatorname{Hom}(A,B)}$.
(f) For composable homomorphisms, $(\alpha'\alpha)^*=(\alpha)^*(\alpha')^*$.
(g) The map $\operatorname{Hom}(\alpha,\beta) : \operatorname{Hom}(A',B)\longrightarrow\operatorname{Hom}(A,B')$ defined by $\operatorname{Hom}(\alpha,\beta)=\alpha^*\beta_*=\beta_*\alpha^*$

$$\big(i.e., \quad A'\xrightarrow{f'}B \quad\longmapsto\quad A\xrightarrow{\alpha}A'\xrightarrow{f'}B\xrightarrow{\beta}B'\big)$$

is a well-defined homomorphism of abelian groups.
(h) $\operatorname{Hom}(\mathrm{id},\mathrm{id})=\mathrm{id}$
(i) For composable homomorphisms, $\operatorname{Hom}(\alpha'\alpha,\beta'\beta)=\operatorname{Hom}(\alpha,\beta')\operatorname{Hom}(\alpha',\beta)$.
(j) $\operatorname{Hom}(\alpha_1+\alpha_2,\beta)=\operatorname{Hom}(\alpha_1,\beta)+\operatorname{Hom}(\alpha_2,\beta)$.
(k) $\operatorname{Hom}(\alpha,\beta_1+\beta_2)=\operatorname{Hom}(\alpha,\beta_1)+\operatorname{Hom}(\alpha,\beta_2)$.

This exercise proves that $\operatorname{Hom}(-,-)$ is an additive functor.

Proposition 4. *Let D be an abelian group.*

(a) *If $0\to A\xrightarrow{\alpha}B\xrightarrow{\beta}C$ is an exact sequence of abelian groups, then*

$$0\to\operatorname{Hom}(D,A)\xrightarrow{\alpha_*}\operatorname{Hom}(D,B)\xrightarrow{\beta_*}\operatorname{Hom}(D,C)$$

is an exact sequence (of abelian groups).

(b) *If $A\xrightarrow{\alpha}B\xrightarrow{\beta}C\to 0$ is an exact sequence of abelian groups, then*

$$0\to\operatorname{Hom}(C,D)\xrightarrow{\beta^*}\operatorname{Hom}(B,D)\xrightarrow{\alpha^*}\operatorname{Hom}(A,D)$$

is an exact sequence (of abelian groups).

Any exact sequence of the form $0\to A\to B\to C$ is called a *left exact sequence*. Proposition 4 asserts that $\operatorname{Hom}(D,-)$ preserves left exact sequences, and that $\operatorname{Hom}(-,D)$ sends right exact sequences to left exact sequences.

Exercise 11. Prove Proposition 4.

Exercise 12. Define $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ for each $m\geq 1$, and let B be an abelian group. Prove the following.

(a) $\mathrm{Hom}(\mathbb{Z}_m, B) \cong B[m] := \{b \in B |\ mb = 0\}$.
(b) $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d$ with $d := \gcd(m, n)$.

To get started on (a), consider the exact sequence $\mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}_m \to 0$. By Proposition 4, applying $\mathrm{Hom}(-, B)$ gives an exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}_m, B) \to \mathrm{Hom}(\mathbb{Z}, B) \to \mathrm{Hom}(\mathbb{Z}, B).$$

By Proposition 4, $\mathrm{Hom}(D, -)$ preserves injective maps and $\mathrm{Hom}(-, D)$ sends surjective maps to injective maps. In general, $\mathrm{Hom}(D, -)$ does not preserve surjective maps, and $\mathrm{Hom}(-, D)$ does not send injective maps to surjective maps.

Exercise 13. Define $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$, and consider the exact sequence

$$0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

(a) Apply $\mathrm{Hom}(\mathbb{Z}_2, -)$ and prove that the resulting map

$$\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2)$$

is not surjective.

(b) Apply $\mathrm{Hom}(-, 2\mathbb{Z})$ and prove that the resulting map

$$\mathrm{Hom}(\mathbb{Z}, 2\mathbb{Z}) \to \mathrm{Hom}(2\mathbb{Z}, 2\mathbb{Z})$$

is not surjective.

Exercise 14. Let A, A', B, B' be abelian groups. Let $\{A_t\}_t$ and $\{B_t\}_t$ be collections of abelian groups indexed on a set. Prove the following.
(a) $\mathrm{Hom}(A \oplus A', B) \cong \mathrm{Hom}(A, B) \oplus \mathrm{Hom}(A', B)$.
(b) $\mathrm{Hom}(A, B \oplus B') \cong \mathrm{Hom}(A, B) \oplus \mathrm{Hom}(A, B')$.
(c) $\mathrm{Hom}(\oplus_t A_t, B) \cong \prod_t \mathrm{Hom}(A_t, B)$.
(d) $\mathrm{Hom}(A, \prod_t B_t) \cong \prod_t \mathrm{Hom}(A, B_t)$.
(e) $\mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z}^n) \cong \mathbb{Z}^{mn}$.

Proposition 5. *Let A, B, C be abelian groups. There are isomorphisms of abelian groups*

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$

natural in A, B, C.

Exercise 15. Prove Proposition 5.

Here are some references for this material: [1, Chapters I and V].

References

[1] S. Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.