Homology and cohomology **Spring 2009**
Prof. Kathryn Hess

Series 10

The purpose of the first part of this series is to show that the integral homology groups of a space X determine the homology and cohomology groups of X with arbitary coefficients. The statements are known as the universal coefficient theorems; one is for homology and involves the Tor functor, the other is for cohomology and involves the Ext functor.

Recall from lecture the following theorem.

Theorem 1 (Universal Coefficient Theorem)**.** *Let C be a chain complex of free abelian groups C_n and let G be any abelian group. Then for each n there is an exact sequence*

$$0 \to H_n(C)\otimes G \xrightarrow{\alpha} H_n(C\otimes G) \xrightarrow{\beta} \mathrm{Tor}(H_{n-1}(C), G) \to 0$$

with homomorphisms α, β natural in C, G. This sequence splits, by a homomorphism which is natural in G but not in C. Here, $\alpha([c]\otimes g) = [c\otimes g]$.

The singular homology version of this theorem follows immediately.

Exercise 1. Observe that Theorem 1 implies Theorem 2.

Theorem 2 (Universal Coefficient Theorem)**.** *Let X be a topological space and let G be any abelian group. Then for each n there is an exact sequence*

$$0 \to H_n(X)\otimes G \xrightarrow{\alpha} H_n(X; G) \xrightarrow{\beta} \mathrm{Tor}(H_{n-1}(X), G) \to 0$$

with homomorphisms α, β natural in X, G. This sequence splits, by a homomorphism which is natural in G but not in X, and hence

$$H_n(X; G) \cong (H_n(X)\otimes G) \oplus \mathrm{Tor}(H_{n-1}(X), G).$$

Here, $\alpha([c]\otimes g) = [c\otimes g]$.

Let $f : X \longrightarrow Y$ be a map of topological spaces. The purpose of the following exercise is to prove that if f induces an isomorphism on integral homology, then f induces an isomorphism on homology with coefficients in any abelian group G.

Exercise 2. Use Theorem 2 together with the Five Lemma to prove Proposition 3.

Proposition 3. *Let $f : X \longrightarrow Y$ be a map of topological spaces and let G be any abelian group. If the induced map $f_* : H_n(X) \xrightarrow{\cong} H_n(Y)$ is an isomorphism for all n, then the induced map*

$$f_* : H_n(X; G) \xrightarrow{\cong} H_n(Y; G)$$

is an isomorphism for all n.

The following characterization [2, V.6.2] of flat abelian groups will be useful in Exercise 3.

Proposition 4. *Let G be an abelian group. Then G is flat if and only if G is torsion-free.*

Exercise 3. Let X be a topological space and let G be a torsion-free abelian group. Use Theorem 2 to prove the following.

(a) $H_n(X;G) \cong H_n(X) \otimes G$ for all $n \geq 0$.
(b) $H_n(X;\mathbb{Q}) \cong H_n(X) \otimes \mathbb{Q}$ for all $n \geq 0$.

The purpose of the next part of this series is to introduce a universal coefficient theorem for cohomology. A first step is to recall from [Series 1] the following exactness property of the Hom functor.

Proposition 5. *Let D be an abelian group. If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is an exact sequence of abelian groups, then*

$$0 \to \mathrm{Hom}(D,A) \xrightarrow{\alpha_*} \mathrm{Hom}(D,B) \xrightarrow{\beta_*} \mathrm{Hom}(D,C)$$

is an exact sequence (of abelian groups).

The purpose of the following exercise is to establish a useful characterization of projective abelian groups.

Exercise 4. Prove Proposition 6.

Proposition 6. *The following properties of an abelian group P are equivalent:*

(i) *P is projective*
(ii) *For each epimorphism $B \xrightarrow{\beta} C$, the induced map $\mathrm{Hom}(P,B) \xrightarrow{\beta_*} \mathrm{Hom}(P,C)$ is an epimorphism.*
(iii) *The functor $\mathrm{Hom}(P,-)$ preserves short exact sequences of abelian groups.*

Let D be an abelian group. Recall that a *resolution* X of D is an exact sequence

$$\cdots \to X_n \xrightarrow{\partial} X_{n-1} \xrightarrow{\partial} \cdots \to X_1 \xrightarrow{\partial} X_0 \xrightarrow{\varepsilon} D \to 0$$

of abelian groups. The chain complex X, which has the form

$$\cdots \to X_n \xrightarrow{\partial} X_{n-1} \xrightarrow{\partial} \cdots \to X_1 \xrightarrow{\partial} X_0 \to 0,$$

is *free* (resp. *projective*) if each X_n is free (resp. projective). Recall from [Series 3] that if D is an abelian group, then applying $\mathrm{Hom}(-,D)$ to the chain complex X gives a cochain complex $\mathrm{Hom}(X,D)$ of the form

$$\cdots \leftarrow \mathrm{Hom}(X_{n+1},D) \xleftarrow{\delta:=(\partial,\mathrm{id})} \mathrm{Hom}(X_n,D) \xleftarrow{\delta:=(\partial,\mathrm{id})} \mathrm{Hom}(X_{n-1},D) \leftarrow \cdots$$

Also recall that every abelian group D has a free resolution F of the form

$$0 \to F_1 \to F_0 \to D \to 0. \tag{1}$$

The following definition of Ext will be needed below.

Definition 7. Let D, G be abelian groups and let F be a free resolution of D of the form (1). Define the abelian group $\mathrm{Ext}(D, G)$ by

$$\mathrm{Ext}(D, G) := \mathrm{Ext}^1(D, G) := H^1(\mathrm{Hom}(F, G)).$$

This definition of $\mathrm{Ext}(D, G)$ involves a particular choice of free resolution F. The following proposition [2, III.6.1] will be needed in Exercise 5 to prove that $\mathrm{Ext}(D, G)$ is independent of the choice of free resolution.

Proposition 8 (Comparison Theorem)**.** *If $\gamma : D \longrightarrow D'$ is a homomorphism of abelian groups, $\varepsilon : X \longrightarrow D$ is a projective resolution of D, and $\varepsilon' : X' \longrightarrow D'$ is a projective resolution of D', then there exists a map of chain complexes $f : X \longrightarrow X'$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon} & D \\ {\scriptstyle \exists\, f}\downarrow & & \downarrow{\scriptstyle \gamma} \\ X' & \xrightarrow{\varepsilon'} & D' \end{array}$$

commutes, and any two such maps are chain homotopic.

Exercise 5. Let D be an abelian group and let $\varepsilon : X \longrightarrow D$ and $\varepsilon' : X' \longrightarrow D$ be projective resolutions of D. Prove the following.

(a) There exist maps f, g of chain complexes which make the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon} & D \\ {\scriptstyle f}\uparrow\ \downarrow{\scriptstyle g} & & \Vert \\ X' & \xrightarrow{\varepsilon'} & D' \end{array}$$

commute; i.e., such that $\varepsilon f = \varepsilon'$ and $\varepsilon' g = \varepsilon$.

(b) fg and gf are chain homotopic to the identity.

(c) If $F : \mathbf{Ab} \longrightarrow \mathbf{Ab}$ is an additive functor, then $(Ff)(Fg)$ and $(Fg)(Ff)$ are chain homotopic to the identity.

(d) $H^n(\mathrm{Hom}(X, G)) \cong H^n(\mathrm{Hom}(X', G))$ for all n.

(e) $\mathrm{Ext}(D, G)$ is independent of the choice of free resolution of D.

The following exercise indicates how the $\mathrm{Ext}(D, -)$ functor provides a measure of the inexactitude of $\mathrm{Hom}(D, -)$.

Exercise 6. Use Proposition 6 to prove Proposition 9.

Proposition 9. *Let D be an abelian group and let F be a free resolution of D of the form* (1)*. Consider any short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of abelian groups. Then there is a short exact sequence*

$$0 \to \mathrm{Hom}(F, A) \xrightarrow{(\alpha, \mathrm{id})} \mathrm{Hom}(F, B) \xrightarrow{(\beta, \mathrm{id})} \mathrm{Hom}(F, C) \to 0$$

of cochain complexes, and hence a corresponding long exact sequence

$$0 \to \mathrm{Hom}(D, A) \to \mathrm{Hom}(D, B) \to \mathrm{Hom}(D, C)$$
$$\xrightarrow{\delta} \mathrm{Ext}(D, A) \to \mathrm{Ext}(D, B) \to \mathrm{Ext}(D, C) \to 0$$

of abelian groups.

The purpose of the following exercise is to prove some of the basic properties of the Ext functor. The properties are sufficient to calculate $\mathrm{Ext}(D, G)$ whenever D and G are finitely generated abelian groups.

Exercise 7. Let D, D', G, G' be abelian groups. Define $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ for each $n \geq 1$. Prove the following.

(a) $\mathrm{Ext}(P, G) = 0$, for all projective P.
(b) $\mathrm{Ext}(D, G \oplus G') \cong \mathrm{Ext}(D, G) \oplus \mathrm{Ext}(D, G')$.
(c) $\mathrm{Ext}(D \oplus D', G) \cong \mathrm{Ext}(D, G) \oplus \mathrm{Ext}(D', G)$.
(d) $\mathrm{Ext}(\mathbb{Z}_m, G) \cong G/mG$.
(e) $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) \cong Z_d$, with $d := \gcd(m, n)$.

To get started on (d), consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}_m \to 0$$

of abelian groups, regarded as a free resolution F of $\mathbb{Z}_m$.

The following theorem, which is proved in Exercise 10, shows that the homology groups of a chain complex C determine the cohomology groups of C with arbitrary coefficients.

Theorem 10 (Universal Coefficient Theorem)**.** *Let C be a chain complex of free abelian groups C_n and let G be any abelian group. Then for each n there is an exact sequence*

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \xrightarrow{\alpha} H^n(\mathrm{Hom}(C, G)) \xrightarrow{\beta} \mathrm{Hom}(H_n(C), G) \to 0$$

with homomorphisms α, β natural in C, G. This sequence splits, by a homomorphism which is natural in G but not in C. Here, $(\beta[f])([c]) = f(c)$.

The singular cohomology version of this theorem follows immediately.

Exercise 8. Observe that Theorem 10 implies Theorem 11.

Theorem 11 (Universal Coefficient Theorem)**.** *Let X be a topological space and let G be any abelian group. Then for each n there is an exact sequence*

$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \xrightarrow{\alpha} H^n(X; G) \xrightarrow{\beta} \mathrm{Hom}(H_n(X), G) \to 0$$

with homomorphisms α, β natural in X, G. This sequence splits, by a homomorphism which is natural in G but not in X, and hence

$$H^n(X; G) \cong \mathrm{Ext}(H_{n-1}(X), G) \oplus \mathrm{Hom}(H_n(X), G).$$

Here, $(\beta[f])([c]) = f(c)$.

Let $f : X \longrightarrow Y$ be a map of topological spaces. The purpose of the following exercise is to prove that if f induces an isomorphism on integral homology, then f induces an isomorphism on cohomology with coefficients in any abelian group G.

Exercise 9. Use Theorem 11 together with the Five Lemma to prove Proposition 12.

Proposition 12. *Let $f : X \longrightarrow Y$ be a map of topological spaces and let G be any abelian group. If the induced map $f_* : H_n(X) \xrightarrow{\cong} H_n(Y)$ is an isomorphism for all n, then the induced map*

$$f^* : H^n(X;G) \xleftarrow{\cong} H^n(Y;G)$$

is an isomorphism for all n.

Exercise 10. Work through the details of a proof of Theorem 10. See the argument in [2, III.4.1] or the argument in [1, V.7.1].

The purpose of the next part of this series is to begin to gain some familiarity with the statement of the Kunneth Theorem, which can be used to calculate the homology of the tensor product $C \otimes C'$, in terms of the homology of C and the homology of C'. We will soon prove the Kunneth theorem together with a corresponding topological version.

Theorem 13 (Kunneth Theorem)**.** *Let C be a chain complex of free abelian groups C_n and let C' be any chain complex. Then for each dimension n there is an exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \xrightarrow{\alpha} H_n(C \otimes C') \xrightarrow{\beta} \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C), H_q(C')) \to 0$$

with homomorphisms α, β natural in C, G. This sequence splits, by a homomorphism which is not natural, and hence

$$H_n(C \otimes C') \cong \Big(\bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \Big) \oplus \Big(\bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C), H_q(C')) \Big).$$

Here, $\alpha([c] \otimes [c']) = [c \otimes c']$.

Exercise 11. Consider the following chain complexes C and C',

$$\begin{array}{lccccccccc} C: & 0 & \longleftarrow & \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} & \longleftarrow & 0 \\ C': & 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 \\ & -1 & & 0 & & 1 & & 2 & & 3 \end{array}$$

(a) Calculate $H_*(C)$.
(b) Calculate $H_*(C')$.
(c) Use Theorem 13 to calculate $H_*(C \otimes C')$.

Exercise 12. Consider the following chain complexes C and C',

$$\begin{array}{lccccccccc} C: & 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \longleftarrow & 0 \\ C': & 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \longleftarrow & 0 \\ & -1 & & 0 & & 1 & & 2 & & 3 \end{array}$$

(a) Calculate $H_*(C)$.
(b) Calculate $H_*(C')$.
(c) Use Theorem 13 to calculate $H_*(C \otimes C')$.

The purpose of the following exercise is to establish a technical proposition that will be used in lecture.

Exercise 13. Prove Proposition 14.

Proposition 14. *If* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$ *is an exact sequence of abelian groups, then*

$$0 \to \operatorname{coker}(f) \to C \to \ker(k) \to 0$$

is an exact sequence (of abelian groups).

Here are some references for this material: [1, Section V.6], [2, Chapter III and Section V.6], [3, Chapter 17].

References

[1] Glen E. Bredon. *Topology and Geometry.* Springer-Verlag, New York, 1993.
[2] S. Mac Lane. *Homology.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.
[3] J. P. May. *A concise course in algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. Available at: `http://www.math.uchicago.edu/~may/` .