**Homology and cohomology** **Spring 2009**
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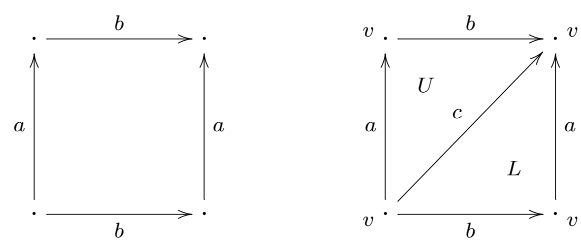
**Series 11**

The purpose of the exercises below is to gain experience in calculating the integral homology groups of some topological spaces directly from a simplicial description, and then to use these calculations together with the Kunneth Theorem and the Universal Coefficient Theorems to obtain additional results.

**Exercise 1.** Recall that the torus $T = S^1 \times S^1$ can be obtained as a quotient of $I^2$ by gluing edges as indicated in the left-hand picture below. In other words, $T \cong I^2/\sim$. Since geometric realization $|-|$ commutes with colimits, it follows easily that $T \cong |X|$, with $X := (\Delta[2] \amalg \Delta[2])/\sim$ the simplicial set obtained by gluing together two copies of the standard simplex $\Delta[2]$ as indicated in the right-hand picture below.




(a) Prove that

$$H_k(T) \cong \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z} \oplus \mathbb{Z}, & k = 1 \\ \mathbb{Z}, & k = 2, \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

by directly calculating the homology of the chain complex $C(X)$.

(b) Give an alternate proof of the calculation (1) by using the Kunneth Theorem together with the calculation

$$H_k(S^1) \cong \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

of the homology groups of $S^1$.

(c) Use the Universal Coefficient Theorems to calculate $H_*(T; \mathbb{Z}_3)$ and $H^*(T; \mathbb{Z}_3)$; here, we define $\mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$.

*Remark* 1. To get started on Exercise 1(a), first note that $C(X)$ has the form

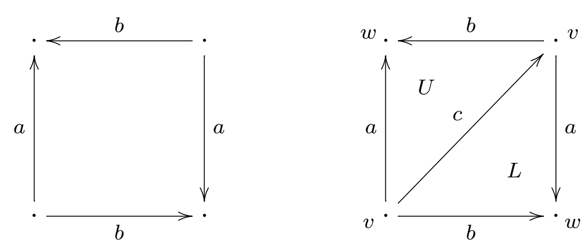
$$C(X): \qquad 0 \xleftarrow{\partial_0} C_0(X) \xleftarrow{\partial_1} C_1(X) \xleftarrow{\partial_2} C_2(X) \xleftarrow{\partial_3} 0$$
$$C_0(X) = \mathbb{Z}\{v\}, \qquad C_1(X) = \mathbb{Z}\{a,b,c\}, \qquad C_2(X) = \mathbb{Z}\{U,L\}$$

Next, calculate the differentials as follows:

$$\begin{aligned}
\partial_1(a) &:= d_0(a) - d_1(a) = v - v = 0, \\
\partial_1(b) &:= d_0(b) - d_1(b) = v - v = 0, \\
\partial_1(c) &:= d_0(c) - d_1(c) = v - v = 0, \\
\partial_2(U) &:= d_0(U) - d_1(U) + d_2(U) = b - c + a, \\
\partial_2(V) &:= d_0(L) - d_1(L) + d_2(L) = a - c + b.
\end{aligned}$$

Next, calculate the kernels of $\partial_k$ and the images of $\partial_k$. Finally, calculate the homology groups $H_k(X) := \ker \partial_k / \operatorname{image} \partial_{k+1}$.

**Exercise 2.** Recall that 2-dimensional real projective space, denoted by $\mathbb{R}P^2$, can be obtained as a quotient of $I^2$ by gluing edges as indicated in the left-hand picture below. In other words, $\mathbb{R}P^2 \cong I^2/\sim$. It follows easily that $\mathbb{R}P^2 \cong |X|$, with $X := (\Delta[2] \amalg \Delta[2])/\sim$ the simplicial set obtained by gluing together two copies of the standard simplex $\Delta[2]$ as indicated in the right-hand picture below.




(a) Prove that

$$H_k(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}/2\mathbb{Z}, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

by directly calculating the homology of the chain complex $C(X)$.

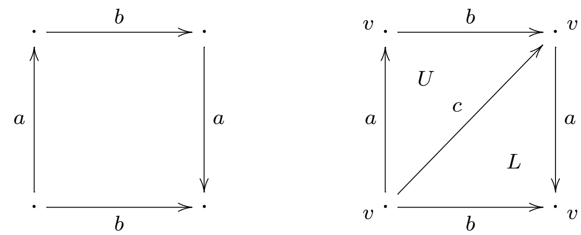
(b) Use the Kunneth Theorem to calculate $H_*(\mathbb{R}P^2 \times \mathbb{R}P^2)$.

(c) Use the Kunneth Theorem to calculate $H_*(\mathbb{R}P^2 \times T)$.

(d) Use the Universal Coefficient Theorem to calculate $H_*(\mathbb{R}P^2; \mathbb{Z}_2)$ and $H_*(\mathbb{R}P^2; \mathbb{Z}_3)$; here, we define $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ for each $n \geq 1$.

**Exercise 3.** Recall that the Klein bottle, denoted by $K$, can be obtained as a quotient of $I^2$ by gluing edges as indicated in the left-hand picture below. In other words, $K \cong I^2/\sim$. It follows easily that $K \cong |X|$, with

$X := (\Delta[2] \amalg \Delta[2])/\sim$ the simplicial set obtained by gluing together two copies of the standard simplex $\Delta[2]$ as indicated in the right-hand picture below.




(a) Prove that

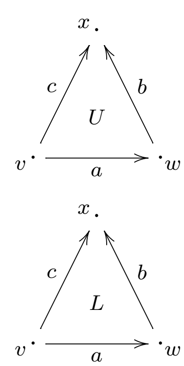
$$H_k(K) \cong \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}), & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

by directly calculating the homology of the chain complex $C(X)$.

(b) Use the Kunneth Theorem to calculate $H_*(K \times \mathbb{R}P^2)$.

(c) Use the Universal Coefficient Theorem to calculate $H^*(K; \mathbb{Z}_2)$ and $H^*(K; \mathbb{Z}_3)$; here, we define $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ for each $n \geq 1$.

**Exercise 4.** Recall that $S^2$ can be obtained by gluing together, along their boundaries, two copies of the standard topological simplex $\Delta^2$. In other words, $S^2 \cong (\Delta^2 \amalg \Delta^2)/\sim$. It follows easily that $S^2 \cong |X|$, with $X := (\Delta[2] \amalg \Delta[2])/\sim$ the simplicial set obtained by gluing together two copies of the standard simplex $\Delta[2]$ as indicated in the picture below.



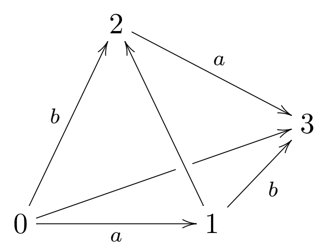

(a) In lecture we calculated $H_*(S^2)$ using the Mayer-Vietoris sequence. Give an alternate proof that

$$H_k(S^2) \cong \begin{cases} \mathbb{Z}, & k = 0, \\ 0, & k = 1, \\ \mathbb{Z}, & k = 2 \\ 0, & \text{otherwise} \end{cases}$$

by directly calculating the homology of the chain complex $C(X)$.

(b) Use the Kunneth Theorem to calculate $H_*(S^2 \times \mathbb{R}P^2)$.

**Exercise 5.** It is easy to check that the torus $T$ is homotopy equivalent to the space $|X|$, with $X := \Delta[3]/\sim$ the simplicial set obtained as a quotient of the standard simplex $\Delta[3]$ as indicated in the picture below.




Hence, it follows from (1) that

$$H_k(|X|) \cong \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z} \oplus \mathbb{Z}, & k = 1 \\ \mathbb{Z}, & k = 2, \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Give an alternate proof of (2) by directly calculating the homology of the chain complex $C(X)$.

*Remark* 2. To get started on Exercise 5, first note that $C(X)$ has the form

$$\begin{array}{llllllllll} C(X): & 0 \xleftarrow{\partial_0} & C_0(X) & \xleftarrow{\partial_1} & C_1(X) & \xleftarrow{\partial_2} & C_2(X) & \xleftarrow{\partial_3} & C_3(X) & \xleftarrow{\partial_4} 0 \\ & & \| & & \| & & \| & & \| & \\ & & \mathbb{Z}\{v\} & & \mathbb{Z}\{a,b,c,d\} & & \mathbb{Z}\{E,F,G,H\} & & \mathbb{Z}\{T\} & \end{array}$$

Here, the non-degenerate simplices are labeled as follows:

$$\begin{array}{lll} v = [0] = [1] = [2] = [3] & E = [0\ 1\ 2] & T = [0\ 1\ 2\ 3] \\ a = [0\ 1] = [2\ 3] & F = [1\ 2\ 3] & \\ b = [1\ 3] = [0\ 2] & G = [0\ 1\ 3] & \\ c = [1\ 2] & H = [0\ 2\ 3] & \\ d = [0\ 3] & & \end{array}$$

Here are some references for this material: [1, Chapter 2].

## References

[1] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. Available at `http://www.math.cornell.edu/~hatcher/` .