Homology and cohomology **Spring 2009**
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Series 12

Definition 1. A chain complex C of abelian groups is of *finite type* if C_n is a finitely generated abelian group for all n.

Theorem 2 (Kunneth Theorem)**.** *Let X and Y be simplicial sets. Assume that $C_*(X)$ is of finite type. Then for each dimension n there is an exact sequence*

$$0 \to \bigoplus_{p+q=n} H^p(X) \otimes H^q(Y) \xrightarrow{\alpha} H^n(X \times Y) \xrightarrow{\beta} \bigoplus_{p+q=n+1} \mathrm{Tor}(H^p(X), H^q(Y)) \to 0$$

with homomorphisms α, β natural in C, G. This sequence splits, by a homomorphism which is not natural, and hence

$$H^n(X \times Y) \cong \Big(\bigoplus_{p+q=n} H^p(X) \otimes H^q(Y) \Big) \oplus \Big(\bigoplus_{p+q=n+1} \mathrm{Tor}(H^p(X), H^q(Y)) \Big)$$

Exercise 1. The purpose of this exercise is to prove Theorem 2, which is a Kunneth Theorem for the cohomology of spaces. Let X be a simplicial set and define $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ for all $n \geq 1$.

(a) Use the algebraic version of the Kunneth Theorem [Series 10, Theorem 13] to prove Theorem 3 below.
(b) Prove that $\mathrm{Hom}(\mathbb{Z}^n, Z) \cong \mathbb{Z}^n$ and $\mathrm{Hom}(\mathbb{Z}_n, Z) = 0$ for all $n \geq 1$.
(c) Use (b) to prove the following: If B is a finitely generated abelian group, then $\mathrm{Hom}(B, \mathbb{Z})$ is a free abelian group.
(d) Prove the following: if $C_*(X)$ is of finite type, then $C^n(X) := \mathrm{Hom}(C_n(X), \mathbb{Z})$ is a free abelian group for all n.
(e) Use Theorem 3 together with (d) to prove Theorem 2.
(f) Observe the following: if X has finitely many non-degenerate n-simplicies for each n, then $C_*(X)$ is of finite type.

To get started on (a), the idea is to regard cochain complexes as chain complexes using the "lower index" notation [Series 3, Remark 4].

Theorem 3 (Kunneth Theorem)**.** *Let C be a cochain complex of free abelian groups C^n and let D be any cochain complex. Then for each dimension n there is an exact sequence*

$$0 \to \bigoplus_{p+q=n} H^p(C) \otimes H^q(D) \xrightarrow{\alpha} H^n(C \otimes D) \xrightarrow{\beta} \bigoplus_{p+q=n+1} \mathrm{Tor}(H^p(C), H^q(D)) \to 0$$

with homomorphisms α, β natural in C, G. This sequence splits, by a homomorphism which is not natural.

Exercise 2. Prove Theorem 4.

Theorem 4. *Let X and Y be simplicial sets. Assume that $C_*(X)$ is of finite type. If $H^*(X)$ or $H^*(Y)$ is torsion-free, then the natural map*

$$H^*(X \times Y) \xleftarrow{\cong} H^*(X) \otimes H^*(Y)$$

is an isomorphism of graded $\mathbb{Z}$-algebras (i.e., graded rings).

Remark 5. To get started on Exercise 2, recall from [Series 10, Proposition 4] that an abelian group G is flat if and only if G is torsion-free.

Exercise 3. Prove Proposition 6.

Proposition 6. *Let X and Y be simplicial sets. Let $\{X_t\}_t$ be a collection of simplicial sets indexed on a set.*

(a) *The natural map*

$$H^*(X \coprod Y) \xrightarrow{\cong} H^*(X) \times H^*(Y)$$

induced by the inclusions $X \to X \amalg Y$ and $Y \to X \amalg Y$ is an isomorphism of graded $\mathbb{Z}$-algebras (i.e., graded rings).

(b) *The natural map*

$$H^*(\textstyle\coprod_t X_t) \xrightarrow{\cong} \textstyle\prod_t H^*(X_t)$$

induced by the inclusions $X_t \to \amalg_t X_t$ is an isomorphism of graded $\mathbb{Z}$-algebras (i.e., graded rings).

In parts (a) and (b), the indicated product is the product in the category of graded rings.

Recall the following from lecture.

Proposition 7. *Let X be a topological space. Denote by $\pi_0(X)$ the set of path components of X. Then*

$$H_0(X) \cong \bigoplus_{\pi_0(X)} \mathbb{Z}$$

the direct sum of $|\pi_0(X)|$ copies of $\mathbb{Z}$. In particular, $H_0(\emptyset) = 0$.

Exercise 4.

(a) Prove Proposition 8 using Proposition 7 together with the Universal Coefficient Theorem for the cohomology of spaces.

(b) Prove that $H^0(S^0 \vee S^0) \cong \mathbb{Z}^3$.

(c) Prove that $H^0(S^0) \oplus H^0(S^0) \cong \mathbb{Z}^4$.

Proposition 8. *Let X be a topological space. Denote by $\pi_0(X)$ the set of path components of X. Then*

$$H^0(X) \cong \prod_{\pi_0(X)} \mathbb{Z}$$

the product of $|\pi_0(X)|$ copies of $\mathbb{Z}$. In particular, $H^0(\emptyset) = 0$.

Let Z be a non-empty simplicial set, consider the natural map $p : Z \longrightarrow *$, and note that $C_*(*) \cong \mathbb{Z}$; here $\mathbb{Z}$ denotes the chain complex concentrated at 0 with value $\mathbb{Z}$. Hence $C_*(Z)$ is an augmented chain complex. The *reduced chain complex* $\tilde{C}_*(Z)$ of the augmented chain complex $C_*(Z)$ is defined by the short exact sequence

$$0 \to \tilde{C}_*(Z) \to C_*(Z) \xrightarrow{p_*} C_*(*) \to 0$$

of chain complexes. The *reduced cohomology* of Z, denoted $\tilde{H}^*(Z)$, is defined by $\tilde{H}^n(Z) := H^n\big(\mathrm{Hom}(\tilde{C}_*(Z), \mathbb{Z})\big)$.

Exercise 5. Let Z be a non-empty simplicial set. Prove the following.

(a) $H^n(Z) \cong \tilde{H}^n(Z)$, for all $n \geq 1$.

(b) There is a short exact sequence

$$0 \leftarrow \tilde{H}^0(Z) \leftarrow H^0(Z) \leftarrow H^0(*) \leftarrow 0$$

of abelian groups.

(c) $H^0(Z) \cong \tilde{H}^0(Z) \oplus \mathbb{Z}$.

(d) $\tilde{H}^n(*) = 0$ for all n.

To get started on (c), note that $*$ is a retract of Z.

The following is a reduced version of the Mayer-Vietoris sequence for cohomology.

Theorem 9 (Reduced Mayer-Vietoris Theorem)**.** *Let Z be a simplicial set and suppose $X, Y \subset Z$ are subcomplexes. Assume that $X \cap Y \neq \emptyset$. Consider the commutative diagram of inclusions*

$$\begin{array}{ccc} X \cap Y & \xrightarrow{i_Y} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle j_Y} \\ X & \xrightarrow[j_X]{} & X \cup Y \end{array}$$

in **sSet**. *Then there is a long exact sequence*

$$\cdots \leftarrow \tilde{H}^n(X \cap Y) \xleftarrow{i_X^* - i_Y^*} \tilde{H}^n(X) \oplus \tilde{H}^n(Y) \xleftarrow{(j_X^*, j_Y^*)} \tilde{H}^n(X \cup Y)$$
$$\xleftarrow{\delta} \tilde{H}^{n-1}(X \cap Y) \leftarrow \cdots \leftarrow \tilde{H}^0(X \cup Y) \leftarrow 0$$

of abelian groups.

Exercise 6. Prove Proposition 10 using the reduced Mayer-Vietoris sequence for cohomology.

Proposition 10. *Let X, Y be based simplicial sets. Then the natural map*

$$\tilde{H}^*(X \vee Y) \xrightarrow{\cong} \tilde{H}^*(X) \times \tilde{H}^*(Y)$$

induced by the inclusions $X \to X \vee Y$ and $Y \to X \vee Y$ is an isomorphism of graded $\mathbb{Z}$-algebras (i.e., graded rings). Here, the indicated product is the product in the category of graded rings.

Exercise 7. Calculate $H^*(\mathrm{N}(\mathbb{Z}_2); \mathbb{F}_2)$ as a graded $\mathbb{Z}$-algebra (i.e., graded ring). Here, we define $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{N}(-)$ denotes the nerve functor [Series 5, Definition 2].

Exercise 8. Let K be a simplicial set.

(a) Work through the details of a proof verifying that the map

$$\Psi : C_*(K) \longrightarrow C_*(K) \otimes C_*(K)$$

is homotopy commutative.

(b) Work through the details of a proof verifying that the cup product

$$H^*(K) \otimes H^*(K) \to H^*(K)$$

is associative and commutative.

For parts (a) and (b), see [1, Chapter VI], [3, Sections VIII.9 and VIII.8], or [4, Chapter 12].

Here are some references for this material: [1, Chapter VI], [2, Chapter 3], [3, Chapter VIII], [4, Chapter 12].

References

[1] Glen E. Bredon. *Topology and Geometry*. Springer-Verlag, New York, 1993.

[2] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. Available at `http://www.math.cornell.edu/~hatcher/` .

[3] S. Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.

[4] Joseph J. Rotman. *An introduction to algebraic topology*, volume 119 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988.