

Series 2

A *preorder* is a set equipped with a reflexive and transitive binary relation; i.e.,

$$\begin{aligned} & \text{(reflexive)} \quad p \leq p, \\ & \text{(transitive)} \quad p \leq p', p' \leq p'' \implies p \leq p'' \end{aligned}$$

for every  $p, p', p''$  in the set.

**Exercise 1.** Prove that a preorder is the same as a category  $\mathbf{P}$  in which, given objects  $p$  and  $p'$ , there is at most one arrow  $p \rightarrow p'$ . The idea is, given such a category, define a binary relation  $\leq$  on the objects of  $\mathbf{P}$  with  $p \leq p'$  if and only if there is a morphism  $p \rightarrow p'$ .

For each  $n \geq 0$ , denote by  $\mathbf{n}$  the category associated to the set  $\{1, \dots, n\}$  equipped with its natural ordering, where  $\mathbf{0}$  denotes the empty category (no objects, no morphisms).

**Exercise 2.** Let  $\mathbf{C}$  be a category.

- (a) Show that the functors  $\mathbf{1} \rightarrow \mathbf{C}$ ,  $\mathbf{2} \rightarrow \mathbf{C}$ , and  $\mathbf{3} \rightarrow \mathbf{C}$  correspond to objects in  $\mathbf{C}$ , morphisms in  $\mathbf{C}$ , and composable pairs of morphisms in  $\mathbf{C}$ , respectively.
- (b) Describe the morphisms of the categories  $\mathbf{C}^{\mathbf{1}}$ ,  $\mathbf{C}^{\mathbf{2}}$ , and  $\mathbf{C}^{\mathbf{3}}$ .

If  $S$  is a set, denote by  $S$  the category with set of objects  $S$  and no non-identity morphisms.

**Exercise 3.** Let  $S$  be a non-empty set. Prove the following.

- (a)  $\mathbf{C}^{\bullet \bullet} = \mathbf{C} \times \mathbf{C}$ ; here,  $\bullet \bullet$  denotes a two element set.
- (b)  $\mathbf{C}^{\bullet \bullet \bullet} = \mathbf{C} \times \mathbf{C} \times \mathbf{C}$ ; here  $\bullet \bullet \bullet$  denotes a three element set.
- (c)  $\mathbf{C}^S = \prod_S \mathbf{C}$ .

Denote by  $\mathbb{N}$  (resp.  $\mathbb{Z}$ ) the category with set of objects  $\{0, 1, 2, \dots\}$  (resp.  $\{0, \pm 1, \pm 2, \dots\}$ ) and no non-identity morphisms. Denote by  $\mathbf{Set}$  the category of sets and their maps, and denote by  $\mathbf{Ab}$  the category of abelian groups and their homomorphisms.

**Exercise 4.** Let  $\mathbf{C}$  be a category.

- (a) Describe the objects and morphisms of the category  $\mathbf{C}^{\mathbb{N}}$ .
- (b) Describe the objects and morphisms of the category  $\mathbf{C}^{\mathbb{Z}}$ .
- (c) Prove that  $\mathbf{Ab}^{\mathbb{N}} = (\mathbb{N}\text{-graded abelian groups})$ .
- (d) Prove that  $\mathbf{Set}^{\mathbb{N}} = (\mathbb{N}\text{-graded sets})$ .
- (e) Prove that  $\mathbf{Ab}^{\mathbb{Z}} = (\mathbb{Z}\text{-graded abelian groups})$ .

Let  $G$  be a group. The category  $\Sigma G$  (denoted here by  $\mathbf{G}$ ) is the category with exactly one object  $\bullet$  and morphisms  $\bullet \xrightarrow{g} \bullet$  the elements  $g$  in  $G$ .

**Exercise 5.** Let  $G$  be a group.

- (a) Show that a functor  $X : \mathbf{G}^{\text{op}} \rightarrow \mathbf{Set}$  corresponds to a set  $X$  equipped with a right  $G$ -action.
- (b) Prove that  $\mathbf{Set}^G$  is the category of left  $G$ -sets and their  $G$ -equivariant maps.
- (c) Prove that  $\mathbf{Set}^{\mathbf{G}^{\text{op}}}$  is the category of right  $G$ -sets and their  $G$ -equivariant maps.
- (d) Prove that  $\mathbf{Ab}^G$  is the category of left  $\mathbb{Z}[G]$ -modules.
- (e) Prove that  $\mathbf{Ab}^{\mathbf{G}^{\text{op}}}$  is the category of right  $\mathbb{Z}[G]$ -modules.

Here,  $\mathbb{Z}[G]$  denotes the integral group ring.

Let  $\mathbf{C}$  be a category and  $A \in \mathbf{C}$ . The *under category*  $A \downarrow \mathbf{C}$  is the category with objects all pairs  $(f, B)$  where  $B \in \mathbf{C}$  and  $f : A \rightarrow B$  is a morphism of  $\mathbf{C}$ , and with morphisms  $h : (f, B) \rightarrow (f', B')$  those morphisms  $h : B \rightarrow B'$  of  $\mathbf{C}$  such that  $hf = f'$ ; i.e., such that the diagram

$$\begin{array}{ccc} A & \xlongequal{\quad} & A \\ f \downarrow & & \downarrow f' \\ B & \xrightarrow{\quad h \quad} & B' \end{array}$$

commutes. Let  $\mathbf{Top}$  denote the category of topological spaces and their continuous maps. Let  $*$  denote a one element set.

**Exercise 6.** Let  $\mathbf{C}$  be a category and  $A \in \mathbf{C}$ .

- (a) Prove that the under category  $A \downarrow \mathbf{C}$  is a well-defined category.
- (b) Prove that  $* \downarrow \mathbf{Set}$  is the category of based sets and basepoint preserving functions.
- (c) Prove that  $* \downarrow \mathbf{Top}$  is the category of based spaces and basepoint preserving continuous maps.
- (d) Similar to part (c), describe the objects and morphisms of  $\mathbb{Z} \downarrow \mathbf{Ab}$ ; here,  $\mathbb{Z}$  denotes the abelian group of integers.

Let  $\mathbf{C}$  be a category and  $B \in \mathbf{C}$ . The *over category*  $\mathbf{C} \downarrow B$  is the category with objects all pairs  $(A, f)$  where  $A \in \mathbf{C}$  and  $f : A \rightarrow B$  is a morphism of  $\mathbf{C}$ , and with morphisms  $g : (A, f) \rightarrow (A', f')$  those morphisms  $g : A \rightarrow A'$  of  $\mathbf{C}$  such that  $f'g = f$ ; i.e., such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad g \quad} & A' \\ f \downarrow & & \downarrow f' \\ B & \xlongequal{\quad} & B \end{array}$$

commutes. Let  $\mathbf{Rng}$  denote the category of rings and their homomorphisms.

**Exercise 7.** Let  $\mathbf{C}$  be a category and  $B \in \mathbf{C}$ .

- (a) Prove that the over category  $\mathbf{C} \downarrow B$  is a well-defined category.
- (b) Prove that  $\mathbf{Set} \downarrow *$  is the category  $\mathbf{Set}$ .

- (c) Prove that  $\mathbf{Top} \downarrow *$  is the category  $\mathbf{Top}$ .
- (d) Prove that  $\mathbf{Rng} \downarrow \mathbb{Z}$  is the category of augmented rings and augmentation preserving ring homomorphisms.

The following construction will be useful when studying adjoint functors. Given categories and functors  $\mathbf{D}' \xrightarrow{T} \mathbf{C} \xleftarrow{S} \mathbf{D}$ , define  $T \downarrow S$  to be the category with objects all triples  $(A, f, B)$  where  $A \in \mathbf{D}'$ ,  $B \in \mathbf{D}$ , and  $f : TA \rightarrow SB$  is a morphism of  $\mathbf{C}$ , and with morphisms  $(A, f, B) \rightarrow (A', f', B')$  all pairs  $(g, h)$  of morphisms  $g : A \rightarrow A'$ ,  $h : B \rightarrow B'$  such that  $f'Tg = (Sh)f$ ; i.e., such that the diagram

$$\begin{array}{ccc} TA & \xrightarrow{Tg} & TA' \\ f \downarrow & & \downarrow f' \\ SB & \xrightarrow{Sh} & SB' \end{array}$$

commutes.

**Exercise 8.** Consider categories and functors  $\mathbf{D}' \xrightarrow{T} \mathbf{C} \xleftarrow{S} \mathbf{D}$ . Let  $A, B \in \mathbf{C}$ .

- (a) Prove that  $T \downarrow S$  is a well-defined category.
- (b) Prove that  $T \downarrow S$  is the category  $A \downarrow \mathbf{C}$  for an appropriate choice of  $T$  and  $S$ .
- (c) Prove that  $T \downarrow S$  is the category  $\mathbf{C} \downarrow B$  for an appropriate choice of  $T$  and  $S$ .
- (d) Consider the categories and functors  $\mathbf{C} \xrightarrow{\text{id}} \mathbf{C} \xleftarrow{\text{id}} \mathbf{C}$ . Prove that  $\text{id} \downarrow \text{id} = \mathbf{C}^2$ .

Here are some references for this material: [1, Chapters I and II].

#### REFERENCES

- [1] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.