Homology and cohomology **Spring 2009**
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Series 2

A *preorder* is a set equipped with a reflexive and transitive binary relation; i.e.,

$$\begin{aligned} &\text{(reflexive)} \quad p \le p, \\ &\text{(transitive)} \quad p \le p',\ p' \le p'' \implies p \le p'' \end{aligned}$$

for every p, p', p'' in the set.

Exercise 1. Prove that a preorder is the same as a category $\mathbf{P}$ in which, given objects p and p', there is at most one arrow $p \to p'$. The idea is, given such a category, define a binary relation $\le$ on the objects of $\mathbf{P}$ with $p \le p'$ if and only if there is a morphism $p \to p'$.

For each $n \ge 0$, denote by $\mathbf{n}$ the category associated to the set $\{1, \cdots, n\}$ equipped with its natural ordering, where $\mathbf{0}$ denotes the empty category (no objects, no morphisms).

Exercise 2. Let $\mathbf{C}$ be a category.

(a) Show that the functors $\mathbf{1} \to \mathbf{C}$, $\mathbf{2} \to \mathbf{C}$, and $\mathbf{3} \to \mathbf{C}$ correspond to objects in $\mathbf{C}$, morphisms in $\mathbf{C}$, and composable pairs of morphisms in $\mathbf{C}$, respectively.

(b) Describe the morphisms of the categories $\mathbf{C}^{\mathbf{1}}$, $\mathbf{C}^{\mathbf{2}}$, and $\mathbf{C}^{\mathbf{3}}$.

If S is a set, denote by S the category with set of objects S and no non-identity morphisms.

Exercise 3. Let S be a non-empty set. Prove the following.

(a) $\mathbf{C}^{\bullet\ \bullet} = \mathbf{C} \times \mathbf{C}$; here, $\bullet\ \bullet$ denotes a two element set.

(b) $\mathbf{C}^{\bullet\ \bullet\ \bullet} = \mathbf{C} \times \mathbf{C} \times \mathbf{C}$; here $\bullet\ \bullet\ \bullet$ denotes a three element set.

(c) $\mathbf{C}^S = \prod_S \mathbf{C}$.

Denote by $\mathbb{N}$ (resp. $\mathbb{Z}$) the category with set of objects $\{0, 1, 2, \cdots\}$ (resp. $\{0, \pm 1, \pm 2, \cdots\}$) and no non-identity morphisms. Denote by $\mathbf{Set}$ the category of sets and their maps, and denote by $\mathbf{Ab}$ the category of abelian groups and their homomorphisms.

Exercise 4. Let $\mathbf{C}$ be a category.

(a) Describe the objects and morphisms of the category $\mathbf{C}^{\mathbb{N}}$.

(b) Describe the objects and morphisms of the category $\mathbf{C}^{\mathbb{Z}}$.

(c) Prove that $\mathbf{Ab}^{\mathbb{N}}$ = ($\mathbb{N}$-graded abelian groups).

(d) Prove that $\mathbf{Set}^{\mathbb{N}}$ = ($\mathbb{N}$-graded sets).

(e) Prove that $\mathbf{Ab}^{\mathbb{Z}}$ = ($\mathbb{Z}$-graded abelian groups).

Let G be a group. The category ΣG (denoted here by $\mathbf{G}$) is the category with exactly one object $\bullet$ and morphisms $\bullet \xrightarrow{g} \bullet$ the elements g in G.

Exercise 5. Let G be a group.

(a) Show that a functor $X : \mathbf{G}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ corresponds to a set X equipped with a right G-action.
(b) Prove that $\mathbf{Set}^{\mathbf{G}}$ is the category of left G-sets and their G-equivariant maps.
(c) Prove that $\mathbf{Set}^{\mathbf{G}^{\mathrm{op}}}$ is the category of right G-sets and their G-equivariant maps.
(d) Prove that $\mathbf{Ab}^{\mathbf{G}}$ is the category of left $\mathbb{Z}[G]$-modules.
(e) Prove that $\mathbf{Ab}^{\mathbf{G}^{\mathrm{op}}}$ is the category of right $\mathbb{Z}[G]$-modules.

Here, $\mathbb{Z}[G]$ denotes the integral group ring.

Let $\mathbf{C}$ be a category and $A \in \mathbf{C}$. The *under category* $A \downarrow \mathbf{C}$ is the category with objects all pairs (f, B) where $B \in \mathbf{C}$ and $f : A \longrightarrow B$ is a morphism of $\mathbf{C}$, and with morphisms $h : (f, B) \longrightarrow (f', B')$ those morphisms $h : B \longrightarrow B'$ of $\mathbf{C}$ such that $hf = f'$; i.e., such that the diagram

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{h} & B' \end{array}$$

commutes. Let **Top** denote the category of topological spaces and their continuous maps. Let $*$ denote a one element set.

Exercise 6. Let $\mathbf{C}$ be a category and $A \in \mathbf{C}$.

(a) Prove that the under category $A \downarrow \mathbf{C}$ is a well-defined category.
(b) Prove that $* \downarrow \mathbf{Set}$ is the category of based sets and basepoint preserving functions.
(c) Prove that $* \downarrow \mathbf{Top}$ is the category of based spaces and basepoint preserving continuous maps.
(d) Similar to part (c), describe the objects and morphisms of $\mathbb{Z} \downarrow \mathbf{Ab}$; here, $\mathbb{Z}$ denotes the abelian group of integers.

Let $\mathbf{C}$ be a category and $B \in \mathbf{C}$. The *over category* $\mathbf{C} \downarrow B$ is the category with objects all pairs (A, f) where $A \in \mathbf{C}$ and $f : A \longrightarrow B$ is a morphism of $\mathbf{C}$, and with morphisms $g : (A, f) \longrightarrow (A', f')$ those morphisms $g : A \longrightarrow A'$ of $\mathbf{C}$ such that $f'g = f$; i.e., such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & = & B \end{array}$$

commutes. Let **Rng** denote the category of rings and their homomorphisms.

Exercise 7. Let $\mathbf{C}$ be a category and $B \in \mathbf{C}$.

(a) Prove that the over category $\mathbf{C} \downarrow B$ is a well-defined category.
(b) Prove that $\mathbf{Set} \downarrow *$ is the category $\mathbf{Set}$.

(c) Prove that $\mathbf{Top} \downarrow *$ is the category $\mathbf{Top}$.
(d) Prove that $\mathbf{Rng} \downarrow \mathbb{Z}$ is the category of augmented rings and augmentation preserving ring homomorphisms.

The following construction will be useful when studying adjoint functors. Given categories and functors $\mathbf{D'} \xrightarrow{T} \mathbf{C} \xleftarrow{S} \mathbf{D}$, define $T \downarrow S$ to be the category with objects all triples (A, f, B) where $A \in \mathbf{D'}$, $B \in \mathbf{D}$, and $f : TA \longrightarrow SB$ is a morphism of $\mathbf{C}$, and with morphisms $(A, f, B) \to (A', f', B')$ all pairs (g, h) of morphisms $g : A \longrightarrow A'$, $h : B \longrightarrow B'$ such that $f'Tg = (Sh)f$; i.e., such that the diagram

$$\begin{array}{ccc} TA & \xrightarrow{Tg} & TA' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ SB & \xrightarrow{Sh} & SB' \end{array}$$

commutes.

Exercise 8. Consider categories and functors $\mathbf{D'} \xrightarrow{T} \mathbf{C} \xleftarrow{S} \mathbf{D}$. Let $A, B \in \mathbf{C}$.

(a) Prove that $T \downarrow S$ is a well-defined category.
(b) Prove that $T \downarrow S$ is the category $A \downarrow \mathbf{C}$ for an appropriate choice of T and S.
(c) Prove that $T \downarrow S$ is the category $\mathbf{C} \downarrow B$ for an appropriate choice of T and S.
(d) Consider the categories and functors $\mathbf{C} \xrightarrow{\text{id}} \mathbf{C} \xleftarrow{\text{id}} \mathbf{C}$. Prove that $\text{id} \downarrow \text{id} \ = \ \mathbf{C^2}$.

Here are some references for this material: [1, Chapters I and II].

References

[1] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.