Homology and cohomology **Spring 2009**

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Series 3

Exercise 1. Consider the following map $f : A \longrightarrow A'$ of chain complexes,

$$\begin{array}{ccccccccccccccc}
A: & \cdots \longrightarrow & \mathbb{Z} & = & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & = & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \longrightarrow & 0 \\
f\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A': & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0
\end{array}$$

(a) Calculate the homology groups $H_*(A)$ and $H_*(A')$.

(b) Show that the induced map $H_*(f) : H_*(A) \xrightarrow{\cong} H_*(A')$ is an isomorphism.

Conclude that non-isomorphic maps of chain complexes may induce isomorphisms on homology.

The following was proved in lecture.

Theorem 1. *Given any short exact sequence* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *of chain complexes, there is a corresponding long exact sequence of homology groups*

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{\alpha_*} H_n(B) \xrightarrow{\beta_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Furthermore, the connecting homomorphism ∂ *is natural with respect to short exact sequences of chain complexes. Here, we are using the notation* $\alpha_* := H_*(\alpha)$ *and* $\beta_* := H_*(\beta)$.

A chain complex A is *acyclic* if $H_*(A) = 0$ (i.e., if $H_n(A) = 0$ for all n). Note that A is acyclic if and only if A is exact as a sequence, hence one can think of homology as a measure of the non-exactness of a chain complex.

Exercise 2. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of chain complexes. Prove the following.

(a) A is acyclic if and only if $\beta_* : H_*(B) \xrightarrow{\cong} H_*(C)$ is an isomorphism.

(b) C is acyclic if and only if $\alpha_* : H_*(A) \xrightarrow{\cong} H_*(B)$ is an isomorphism.

(c) B is acyclic if and only if $\partial : H_*(C) \xrightarrow{\cong} H_*(A)$ is an isomorphism.

In particular, conclude that if any two of the three chain complexes A, B, C are acyclic, then so is the third.

Exercise 3. Consider any exact sequence of abelian groups

$$\cdots \to A_{n+1} \to A_n \to A_{n-1} \to \cdots$$

which we will denote by A. Let $A' \subset A$ be a subcomplex

$$\cdots \to A'_{n+1} \to A'_n \to A'_{n-1} \to \cdots$$

of A. Use Exercise 2 to prove the following.

(a) The subcomplex A' is exact if and only if the quotient complex A/A'

$$\cdots \to A_{n+1}/A'_{n+1} \to A_n/A'_n \to A_{n-1}/A'_{n-1} \to \cdots$$

is exact.

A map of chain complexes $f : A \longrightarrow A'$ is a *homology isomorphism* if the induced map $H_*(f) : H_*(A) \xrightarrow{\cong} H_*(B)$ on homology is an isomorphism. Recall the following.

Proposition 2 (The Five Lemma)**.** *Consider any commutative diagram in abelian groups of the form*

$$\begin{array}{ccccccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}$$

with exact rows. If f_1, f_2, f_4, f_5 are isomorphisms, then f_3 is an isomorphism. More precisely,

(a) *If f_1 is an epimorphism and f_2, f_4 are monomorphisms, then f_3 is a monomorphism.*
(b) *If f_2, f_4 are epimorphisms and f_5 is a monomorphism, then f_3 is an epimorphism.*

Exercise 4. Consider any commutative diagram in chain complexes of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \xrightarrow[\alpha']{} & B' & \xrightarrow[\beta']{} & C' & \longrightarrow & 0 \end{array}$$

with exact rows. Prove the following.

(a) If $H_*(f)$ and $H_*(g)$ are isomorphisms, then $H_*(h)$ is an isomorphism.
(b) If $H_*(g)$ and $H_*(h)$ are isomorphisms, then $H_*(f)$ is an isomorphism.
(c) If $H_*(f)$ and $H_*(h)$ are isomorphisms, then $H_*(g)$ is an isomorphism.

Conclude that if any two of the three maps f, g, h are homology isomorphisms, then so is the third.

Recall that a *chain homotopy* s between two maps $f, g : A \longrightarrow A'$ of chain complexes, sometimes denoted by $s : f \simeq g$, is a family of homomorphisms $s_n : A_n \longrightarrow A'_{n+1}$, one for each n,

(1)
$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \longrightarrow & \cdots \\ & & {\scriptstyle f}\downarrow\downarrow{\scriptstyle g} & \swarrow{\scriptstyle s} & {\scriptstyle f}\downarrow\downarrow{\scriptstyle g} & \swarrow{\scriptstyle s} & {\scriptstyle f}\downarrow\downarrow{\scriptstyle g} & & \\ \cdots & \longrightarrow & A'_{n+1} & \xrightarrow[\partial]{} & A'_n & \xrightarrow[\partial]{} & A'_{n-1} & \longrightarrow & \cdots \end{array}$$

such that

$$\partial s + s\partial = f - g.$$

Here we have omitted the subscripts from the notation; it is easy to find them from diagram (1).

Remark 3. It is easy to check that the chain homotopy relation $\simeq$ is an equivalence relation on the set of all maps of chain complexes $f : A \longrightarrow A'$. The equivalence classes are called *chain homotopy classes*, and the set of all homotopy classes is sometimes denoted by $[A, A']$. Recall from lecture that $f \simeq g$ implies $H_*(f) = H_*(g)$; in other words, homology is chain homotopy invariant.

Exercise 5. Consider the following maps

$$A \underset{g}{\overset{f}{\rightrightarrows}} A' \underset{g'}{\overset{f'}{\rightrightarrows}} A''$$

of chain complexes. Prove the following.

(a) If $s : f \simeq g$ and $s' : f' \simeq g'$, then $f's + s'g : f'f \simeq g'g$.

Conclude that composites of chain homotopic maps are chain homotopic.

Our next step is to introduce cochain complexes and cohomology. Let A be a chain complex and consider any abelian group G. Then $A \otimes G$ is a chain complex of the form

$$\cdots \to A_{n+1} \otimes G \xrightarrow{\partial \otimes \mathrm{id}} A_n \otimes G \xrightarrow{\partial \otimes \mathrm{id}} A_{n-1} \otimes G \to \cdots$$

Recall from lecture that the *homology of A with coefficients in G* is defined by $H_*(A; G) := H_*(A \otimes G)$. Similarly, applying $\mathrm{Hom}(-, G)$ to the chain complex A gives a sequence of abelian groups of the form

$$\cdots \leftarrow \mathrm{Hom}(A_{n+1}, G) \xleftarrow{\delta := (\partial, \mathrm{id})} \mathrm{Hom}(A_n, G) \xleftarrow{\delta := (\partial, \mathrm{id})} \mathrm{Hom}(A_{n-1}, G) \leftarrow \cdots$$

and it is easy to check that $\delta\delta = 0$. Hence $\mathrm{Hom}(A, G)$ looks like a chain complex except that the maps δ have degree $+1$ instead of degree -1. This example motivates the following definitions.

A *cochain complex* A is a family $\{A^n, \delta^n\}$ of abelian groups and abelian group homomorphisms $\delta^n : A^n \longrightarrow A^{n+1}$, defined for all integers $n \in \mathbb{Z}$,

$$\cdots \leftarrow A^{n+1} \xleftarrow{\delta} A^n \xleftarrow{\delta} A^{n-1} \leftarrow \cdots$$

and such that $\delta\delta = 0$; here we have omitted the superscripts on δ from the notation. The maps δ are called the *differentials* (or coboundary operators). *Maps of cochain complexes* are defined analogously to maps of chain complexes. Given a cochain complex A, the *cohomology of A* is defined by $H^*(A) := \ker \delta / \operatorname{image} \delta$; more precisely,

$$H^n(A) := \ker \delta^n / \operatorname{image} \delta^{n-1}, \quad n \in \mathbb{Z}.$$

Sometimes the elements of A^n are called *n-cochains*, the elements of $\ker \delta^n$ are called *n-cocycles*, and the elements of image δ^{n-1} are called *n-coboundaries*. A *cochain homotopy* s between two maps $f, g : A \longrightarrow A'$ of cochain complexes, sometimes denoted by $s : f \simeq g$, is a family of homomorphisms $s^n : A^n \longrightarrow A'^{n-1}$, one for each n,

(2)
$$\begin{array}{ccccccccc}
\cdots & \longleftarrow & A^{n+1} & \overset{\delta}{\longleftarrow} & A^n & \overset{\delta}{\longleftarrow} & A^{n-1} & \longleftarrow & \cdots \\
 & & f \Big\downarrow\Big\downarrow g & \overset{s}{\searrow} & f \Big\downarrow\Big\downarrow g & \overset{s}{\searrow} & f \Big\downarrow\Big\downarrow g & & \\
\cdots & \longleftarrow & A'^{n+1} & \underset{\delta}{\longleftarrow} & A'^n & \underset{\delta}{\longleftarrow} & A'^{n-1} & \longleftarrow & \cdots
\end{array}$$

such that

$$\delta s + s\delta = f - g.$$

Here we have omitted the subscripts from the notation; it is easy to find them from diagram (2).

Remark 4. Every chain complex $\{A_n, \partial_n\}$ determines a cochain complex $\{A^n, \delta^n\}$ defined by $A^n := A_{-n}$ and $\delta^n := \partial_{-n}$. This is "upper index" notation for the same complex, and it is easy to check that $H^n(A) = H_{-n}(A)$. Conversely, every cochain complex determines a chain complex. In this way, results about chain complexes give results about cochain complexes. For example, $f \simeq g$ implies $H^*(f) = H^*(g)$; in other words, cohomology is cochain homotopy invariant.

Let A be a chain complex and consider any abelian group G. The *cohomology of A with coefficients in G* is defined by $H^*(A; G) := H^*(\mathrm{Hom}(A, G))$; more precisely,

$$H^n(A; G) := H^n(\mathrm{Hom}(A, G)), \quad n \in \mathbb{Z}.$$

Exercise 6. Define $\mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$. Consider the following chain complexes A and B,

$$\begin{array}{ll}
A: & \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{3} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \\
B: & \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots
\end{array}$$

(a) Calculate the homology groups $H_*(A)$ and $H_*(B)$.
(b) Calculate the homology groups $H_*(A; \mathbb{Z}_3)$ and $H_*(B; \mathbb{Z}_3)$.
(c) Calculate the cohomology groups $H^*(A; \mathbb{Z})$ and $H^*(B; \mathbb{Z})$.
(d) Calculate the cohomology groups $H^*(A; \mathbb{Z}_3)$ and $H^*(B; \mathbb{Z}_3)$.

Conclude that homology and cohomology depend on the coefficients; note that $H_*(A) \cong H_*(A; \mathbb{Z})$ for every chain complex A.

Denote by **Ab** the category of abelian groups and their homomorphisms. A functor $F : \mathbf{Ab} \longrightarrow \mathbf{Ab}$ (resp. $F : \mathbf{Ab}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$) is *additive* if for every pair of homomorphisms $f, g : X \longrightarrow Y$ of abelian groups, the following

$$F(f + g) = F(f) + F(g)$$

is satisfied. In Series 1 we proved that for any abelian group G, the functors

$$\mathbf{Ab} \xrightarrow{G\otimes -} \mathbf{Ab}, \quad \mathbf{Ab} \xrightarrow{-\otimes G} \mathbf{Ab},$$

$$\mathbf{Ab} \xrightarrow{\mathrm{Hom}(G,-)} \mathbf{Ab}, \quad \mathbf{Ab}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}(-,G)} \mathbf{Ab}$$

are additive functors.

Exercise 7. Let $F : \mathbf{Ab} \longrightarrow \mathbf{Ab}$ be an additive functor of abelian groups. Consider any maps $f, g : A \longrightarrow A'$ of chain complexes and let G be an abelian group. Prove the following.

(a) If $s : f \simeq g$, then $Fs : Ff \simeq Fg$. In other words, F preserves chain homotopies.
(b) If $f \simeq g$, then $H_*(Ff) = H_*(Fg)$.
(c) If $f \simeq g$, then $H_*(f; G) = H_*(g; G)$. In other words, homology with coefficients in G is chain homotopy invariant.
(d) If $f \simeq g$, then $H^*(f; G) = H^*(g; G)$. In other words, cohomology with coefficients in G is chain homotopy invariant.

Exercise 8. Define $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ for each $n \geq 1$. Consider the following map $f : A \longrightarrow A'$ of chain complexes,

$$\begin{array}{lccccccccccc}
A: & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
{\scriptstyle f}\big\downarrow & & & \big\downarrow & & \big\| & & \big\downarrow & & \big\downarrow & & \\
A': & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_3 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

(a) Calculate the homology groups $H_*(A)$ and $H_*(A')$.
(b) Show that the induced map $H_*(f) = 0 = H_*(0)$.
(c) Calculate the homology groups $H_*(A; \mathbb{Z}_2)$ and $H_*(A'; \mathbb{Z}_2)$.
(d) Show that the induced map $H_*(f; \mathbb{Z}_2) \neq 0 = H_*(0; \mathbb{Z}_2)$.
(e) Use Exercise 7 to prove that f is not chain homotopic to the zero map $0 : A \longrightarrow A'$.

Conclude that: $H_*(f) = H_*(g)$ does not imply $f \simeq g$.

Here are some references for this material: [1, Chapter II], [2, Chapters 4,5]

References

[1] S. Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.

[2] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York, 1981. Corrected reprint.