

Series 3

**Exercise 1.** Consider the following map  $f : A \rightarrow A'$  of chain complexes,

$$\begin{array}{ccccccccccccccc}
 A : & & \cdots & \longrightarrow & \mathbb{Z} & \xlongequal{\quad} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xlongequal{\quad} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \longrightarrow & 0 \\
 f \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 A' : & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0
 \end{array}$$

- (a) Calculate the homology groups  $H_*(A)$  and  $H_*(A')$ .
- (b) Show that the induced map  $H_*(f) : H_*(A) \xrightarrow{\cong} H_*(A')$  is an isomorphism.

Conclude that non-isomorphic maps of chain complexes may induce isomorphisms on homology.

The following was proved in lecture.

**Theorem 1.** Given any short exact sequence  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$  of chain complexes, there is a corresponding long exact sequence of homology groups

$$\cdots \rightarrow H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{\alpha_*} H_n(B) \xrightarrow{\beta_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \cdots$$

Furthermore, the connecting homomorphism  $\partial$  is natural with respect to short exact sequences of chain complexes. Here, we are using the notation  $\alpha_* := H_*(\alpha)$  and  $\beta_* := H_*(\beta)$ .

A chain complex  $A$  is *acyclic* if  $H_*(A) = 0$  (i.e., if  $H_n(A) = 0$  for all  $n$ ). Note that  $A$  is acyclic if and only if  $A$  is exact as a sequence, hence one can think of homology as a measure of the non-exactness of a chain complex.

**Exercise 2.** Let  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$  be a short exact sequence of chain complexes. Prove the following.

- (a)  $A$  is acyclic if and only if  $\beta_* : H_*(B) \xrightarrow{\cong} H_*(C)$  is an isomorphism.
- (b)  $C$  is acyclic if and only if  $\alpha_* : H_*(A) \xrightarrow{\cong} H_*(B)$  is an isomorphism.
- (c)  $B$  is acyclic if and only if  $\partial : H_*(C) \xrightarrow{\cong} H_*(A)$  is an isomorphism.

In particular, conclude that if any two of the three chain complexes  $A, B, C$  are acyclic, then so is the third.

**Exercise 3.** Consider any exact sequence of abelian groups

$$\cdots \rightarrow A_{n+1} \rightarrow A_n \rightarrow A_{n-1} \rightarrow \cdots$$

which we will denote by  $A$ . Let  $A' \subset A$  be a subcomplex

$$\cdots \rightarrow A'_{n+1} \rightarrow A'_n \rightarrow A'_{n-1} \rightarrow \cdots$$

of  $A$ . Use Exercise 2 to prove the following.

(a) The subcomplex  $A'$  is exact if and only if the quotient complex  $A/A'$

$$\cdots \rightarrow A_{n+1}/A'_{n+1} \rightarrow A_n/A'_n \rightarrow A_{n-1}/A'_{n-1} \rightarrow \cdots$$

is exact.

A map of chain complexes  $f : A \rightarrow A'$  is a *homology isomorphism* if the induced map  $H_*(f) : H_*(A) \xrightarrow{\cong} H_*(A')$  on homology is an isomorphism. Recall the following.

**Proposition 2** (The Five Lemma). *Consider any commutative diagram in abelian groups of the form*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \end{array}$$

with exact rows. If  $f_1, f_2, f_4, f_5$  are isomorphisms, then  $f_3$  is an isomorphism. More precisely,

- (a) If  $f_1$  is an epimorphism and  $f_2, f_4$  are monomorphisms, then  $f_3$  is a monomorphism.
- (b) If  $f_2, f_4$  are epimorphisms and  $f_5$  is a monomorphism, then  $f_3$  is an epimorphism.

**Exercise 4.** Consider any commutative diagram in chain complexes of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \longrightarrow & 0 \end{array}$$

with exact rows. Prove the following.

- (a) If  $H_*(f)$  and  $H_*(g)$  are isomorphisms, then  $H_*(h)$  is an isomorphism.
- (b) If  $H_*(g)$  and  $H_*(h)$  are isomorphisms, then  $H_*(f)$  is an isomorphism.
- (c) If  $H_*(f)$  and  $H_*(h)$  are isomorphisms, then  $H_*(g)$  is an isomorphism.

Conclude that if any two of the three maps  $f, g, h$  are homology isomorphisms, then so is the third.

Recall that a *chain homotopy*  $s$  between two maps  $f, g : A \rightarrow A'$  of chain complexes, sometimes denoted by  $s : f \simeq g$ , is a family of homomorphisms  $s_n : A_n \rightarrow A'_{n+1}$ , one for each  $n$ ,

$$(1) \quad \begin{array}{ccccccccc} \cdots & \longrightarrow & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \longrightarrow & \cdots \\ & & \downarrow f & \searrow s & \downarrow f & \searrow s & \downarrow f \\ & & A'_{n+1} & \xrightarrow{\partial} & A'_n & \xrightarrow{\partial} & A'_{n-1} & \longrightarrow & \cdots \end{array}$$

such that

$$\partial s + s\partial = f - g.$$

Here we have omitted the subscripts from the notation; it is easy to find them from diagram (1).

*Remark 3.* It is easy to check that the chain homotopy relation  $\simeq$  is an equivalence relation on the set of all maps of chain complexes  $f : A \rightarrow A'$ . The equivalence classes are called *chain homotopy classes*, and the set of all homotopy classes is sometimes denoted by  $[A, A']$ . Recall from lecture that  $f \simeq g$  implies  $H_*(f) = H_*(g)$ ; in other words, homology is chain homotopy invariant.

**Exercise 5.** Consider the following maps

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} A' \begin{array}{c} \xrightarrow{f'} \\ \xrightarrow{g'} \end{array} A''$$

of chain complexes. Prove the following.

(a) If  $s : f \simeq g$  and  $s' : f' \simeq g'$ , then  $f's + s'g : f'f \simeq g'g$ .

Conclude that composites of chain homotopic maps are chain homotopic.

Our next step is to introduce cochain complexes and cohomology. Let  $A$  be a chain complex and consider any abelian group  $G$ . Then  $A \otimes G$  is a chain complex of the form

$$\cdots \rightarrow A_{n+1} \otimes G \xrightarrow{\partial \otimes \text{id}} A_n \otimes G \xrightarrow{\partial \otimes \text{id}} A_{n-1} \otimes G \rightarrow \cdots$$

Recall from lecture that the *homology of  $A$  with coefficients in  $G$*  is defined by  $H_*(A; G) := H_*(A \otimes G)$ . Similarly, applying  $\text{Hom}(-, G)$  to the chain complex  $A$  gives a sequence of abelian groups of the form

$$\cdots \leftarrow \text{Hom}(A_{n+1}, G) \xleftarrow{\delta := (\partial, \text{id})} \text{Hom}(A_n, G) \xleftarrow{\delta := (\partial, \text{id})} \text{Hom}(A_{n-1}, G) \leftarrow \cdots$$

and it is easy to check that  $\delta\delta = 0$ . Hence  $\text{Hom}(A, G)$  looks like a chain complex except that the maps  $\delta$  have degree  $+1$  instead of degree  $-1$ . This example motivates the following definitions.

A *cochain complex*  $A$  is a family  $\{A^n, \delta^n\}$  of abelian groups and abelian group homomorphisms  $\delta^n : A^n \rightarrow A^{n+1}$ , defined for all integers  $n \in \mathbb{Z}$ ,

$$\cdots \leftarrow A^{n+1} \xleftarrow{\delta} A^n \xleftarrow{\delta} A^{n-1} \leftarrow \cdots$$

and such that  $\delta\delta = 0$ ; here we have omitted the superscripts on  $\delta$  from the notation. The maps  $\delta$  are called the *differentials* (or coboundary operators). *Maps of cochain complexes* are defined analogously to maps of chain complexes. Given a cochain complex  $A$ , the *cohomology of  $A$*  is defined by  $H^*(A) := \ker \delta / \text{image } \delta$ ; more precisely,

$$H^n(A) := \ker \delta^n / \text{image } \delta^{n-1}, \quad n \in \mathbb{Z}.$$

Sometimes the elements of  $A^n$  are called  $n$ -cochains, the elements of  $\ker \delta^n$  are called  $n$ -cocycles, and the elements of image  $\delta^{n-1}$  are called  $n$ -coboundaries. A cochain homotopy  $s$  between two maps  $f, g : A \rightarrow A'$  of cochain complexes, sometimes denoted by  $s : f \simeq g$ , is a family of homomorphisms  $s^n : A^n \rightarrow A'^{n-1}$ , one for each  $n$ ,

$$(2) \quad \begin{array}{ccccccc} \cdots & \longleftarrow & A^{n+1} & \xleftarrow{\delta} & A^n & \xleftarrow{\delta} & A^{n-1} & \longleftarrow & \cdots \\ & & f \downarrow & & \downarrow g & & f \downarrow & & \downarrow g \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & s & & s & & s & & \\ & & \searrow & & \searrow & & \searrow & & \\ \cdots & \longleftarrow & A'^{n+1} & \xleftarrow{\delta} & A'^n & \xleftarrow{\delta} & A'^{n-1} & \longleftarrow & \cdots \end{array}$$

such that

$$\delta s + s \delta = f - g.$$

Here we have omitted the subscripts from the notation; it is easy to find them from diagram (2).

*Remark 4.* Every chain complex  $\{A_n, \partial_n\}$  determines a cochain complex  $\{A^n, \delta^n\}$  defined by  $A^n := A_{-n}$  and  $\delta^n := \partial_{-n}$ . This is “upper index” notation for the same complex, and it is easy to check that  $H^n(A) = H_{-n}(A)$ . Conversely, every cochain complex determines a chain complex. In this way, results about chain complexes give results about cochain complexes. For example,  $f \simeq g$  implies  $H^*(f) = H^*(g)$ ; in other words, cohomology is cochain homotopy invariant.

Let  $A$  be a chain complex and consider any abelian group  $G$ . The cohomology of  $A$  with coefficients in  $G$  is defined by  $H^*(A; G) := H^*(\text{Hom}(A, G))$ ; more precisely,

$$H^n(A; G) := H^n(\text{Hom}(A, G)), \quad n \in \mathbb{Z}.$$

**Exercise 6.** Define  $\mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$ . Consider the following chain complexes  $A$  and  $B$ ,

$$\begin{array}{l} A : \quad \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{3} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \\ B : \quad \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \end{array}$$

- Calculate the homology groups  $H_*(A)$  and  $H_*(B)$ .
- Calculate the homology groups  $H_*(A; \mathbb{Z}_3)$  and  $H_*(B; \mathbb{Z}_3)$ .
- Calculate the cohomology groups  $H^*(A; \mathbb{Z})$  and  $H^*(B; \mathbb{Z})$ .
- Calculate the cohomology groups  $H^*(A; \mathbb{Z}_3)$  and  $H^*(B; \mathbb{Z}_3)$ .

Conclude that homology and cohomology depend on the coefficients; note that  $H_*(A) \cong H_*(A; \mathbb{Z})$  for every chain complex  $A$ .

Denote by  $\mathbf{Ab}$  the category of abelian groups and their homomorphisms. A functor  $F : \mathbf{Ab} \rightarrow \mathbf{Ab}$  (resp.  $F : \mathbf{Ab}^{\text{op}} \rightarrow \mathbf{Ab}$ ) is *additive* if for every pair of homomorphisms  $f, g : X \rightarrow Y$  of abelian groups, the following

$$F(f + g) = F(f) + F(g)$$

is satisfied. In Series 1 we proved that for any abelian group  $G$ , the functors

$$\begin{aligned} \mathbf{Ab} &\xrightarrow{G \otimes -} \mathbf{Ab}, & \mathbf{Ab} &\xrightarrow{- \otimes G} \mathbf{Ab}, \\ \mathbf{Ab} &\xrightarrow{\text{Hom}(G, -)} \mathbf{Ab}, & \mathbf{Ab}^{\text{op}} &\xrightarrow{\text{Hom}(-, G)} \mathbf{Ab} \end{aligned}$$

are additive functors.

**Exercise 7.** Let  $F : \mathbf{Ab} \rightarrow \mathbf{Ab}$  be an additive functor of abelian groups. Consider any maps  $f, g : A \rightarrow A'$  of chain complexes and let  $G$  be an abelian group. Prove the following.

- If  $s : f \simeq g$ , then  $Fs : Ff \simeq Fg$ . In other words,  $F$  preserves chain homotopies.
- If  $f \simeq g$ , then  $H_*(Ff) = H_*(Fg)$ .
- If  $f \simeq g$ , then  $H_*(f; G) = H_*(g; G)$ . In other words, homology with coefficients in  $G$  is chain homotopy invariant.
- If  $f \simeq g$ , then  $H^*(f; G) = H^*(g; G)$ . In other words, cohomology with coefficients in  $G$  is chain homotopy invariant.

**Exercise 8.** Define  $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$  for each  $n \geq 1$ . Consider the following map  $f : A \rightarrow A'$  of chain complexes,

$$\begin{array}{ccccccccccc} A : & & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ f \downarrow & & & & \downarrow & & \parallel & & \downarrow & & \downarrow & & \\ A' : & & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_3 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

- Calculate the homology groups  $H_*(A)$  and  $H_*(A')$ .
- Show that the induced map  $H_*(f) = 0 = H_*(0)$ .
- Calculate the homology groups  $H_*(A; \mathbb{Z}_2)$  and  $H_*(A'; \mathbb{Z}_2)$ .
- Show that the induced map  $H_*(f; \mathbb{Z}_2) \neq 0 = H_*(0; \mathbb{Z}_2)$ .
- Use Exercise 7 to prove that  $f$  is not chain homotopic to the zero map  $0 : A \rightarrow A'$ .

Conclude that:  $H_*(f) = H_*(g)$  does not imply  $f \simeq g$ .

Here are some references for this material: [1, Chapter II], [2, Chapters 4,5]

#### REFERENCES

- [1] S. Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.
- [2] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York, 1981. Corrected reprint.