Homology and cohomology **Spring 2009**

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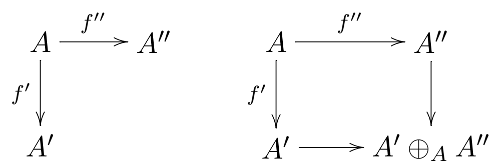
Series 4

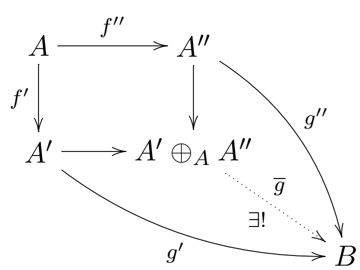
It will be useful to generalize the direct sum and quotient constructions for chain complexes. Denote by **Ch** the category of chain complexes and their maps. Suppose $f' : A \longrightarrow A'$ and $f'' : A \longrightarrow A''$ are maps of chain complexes. We can organize this data into the left-hand diagram

(1)



in **Ch**. The *pushout* (or colimit) of the left-hand diagram, written $A' \oplus_A A''$ or $A' \amalg_A A''$, is a chain complex, together with the right-hand commutative diagram of chain complexes in (1), which satisfies the universal property: given two maps g', g'' of chain complexes such that the outer diagram in (2) commutes, then there exists a unique map $\overline{g}$ of chain complexes such that the diagram

(2)



in **Ch** commutes.

Remark 1. It is easy to verify that pushouts in chain complexes exist; define $A' \oplus_A A'' := A' \oplus A''/\sim$, the direct sum of chain complexes A' and A'' modulo $f'(a) \sim f''(a)$, $a \in A$.

When the maps f', f'' in (2) are monomorphisms, we will sometimes use the "union" notation

$$A' \cup A'' := A' \cup_A A'' := A' \oplus_A A''$$

to denote the indicated pushout. This notation is explained by the following.

Exercise 1. Let X be a chain complex and suppose $A', A'' \subseteq X$ are subcomplexes such that X is the union of A' and A''.

(a) Prove that the diagram of inclusions

$$\begin{CD} A' \cap A'' @>{\subseteq}>> A'' \\ @V{\subseteq}VV @VV{\subseteq}V \\ A' @>{\subseteq}>> X \end{CD}$$

in **Ch** is a pushout diagram.

Note, it is enough to verify the universal property of pushouts.

Exercise 2. Let X be a chain complex and suppose $A \subseteq X$ is a subcomplex.

(a) Prove that the diagram

$$\begin{CD} A @>>> 0 \\ @V{\subseteq}VV @VVV \\ X @>>> X/A \end{CD}$$

in **Ch** is a pushout diagram. Conclude that $X \oplus_A 0 \cong X/A$.

Here, 0 denotes the trivial chain complex and X/A is the quotient complex. Note, it is enough to verify the universal property of pushouts.

Exercise 3. Denote by 0 the trivial chain complex.

(a) Prove that $A' \oplus_A A'' \cong A'' \oplus_A A'$.
(b) Prove that $A' \oplus_0 A'' \cong A' \oplus A''$.

Exercise 4. Consider the left-hand diagram

(3)
$$\begin{array}{ccc} A & \xrightarrow{f''} & A'' \\ {\scriptstyle f'}\downarrow{\scriptstyle \cong} & & \\ A & & \end{array} \qquad \begin{array}{ccccc} A & & \xrightarrow{f''} & & A'' \\ {\scriptstyle f'}\downarrow{\scriptstyle \cong} & & & & \| \\ A & \xrightarrow[f'^{-1}]{\cong} & A & \xrightarrow[f'']{} & A'' \end{array}$$

in **Ch** such that f' is an isomorphism.

(a) Prove that $A \oplus_A A'' \cong A''$.

Note, it is enough to verify that the right-hand diagram in (3) satisfies the universal property of pushouts.

Exercise 5. Let A, B be chain complexes. Let $\{A_t\}_t$ be a collection of chain complexes indexed on a set. Let $n \in \mathbb{Z}$ be any integer. Prove the following.

(a) $H_n(A \oplus B) \cong H_n(A) \oplus H_n(B)$.
(b) $H_n(\oplus_t A_t) \cong \oplus_t H_n(A_t)$.
(c) $H_n(\prod_t A_t) \cong \prod_t H_n(A_t)$.

In other words, homology $H_*(-)$ commutes with arbitrary direct sums and arbitrary products.

The following was proved in lecture.

Proposition 2. *Let X be a chain complex and suppose $A', A'' \subseteq X$ are subcomplexes such that $X = A' \cup A''$; i.e., such that the diagram of inclusions*

$$\begin{array}{ccc} A' \cap A'' & \xrightarrow{i''} & A'' \\ {\scriptstyle i'}\big\downarrow & & \big\downarrow{\scriptstyle j''} \\ A' & \xrightarrow[j']{} & X \end{array}$$

in **Ch** *is a pushout diagram. Then there is a short exact sequence*

$$0 \to A' \cap A'' \xrightarrow{(i', i'')} A' \oplus A'' \xrightarrow{j' - j''} X \to 0$$

of chain complexes, and hence a corresponding long exact sequence

$$\begin{aligned} \cdots \to & H_n(A' \cap A'') \xrightarrow{(i'_*, i''_*)} H_n(A') \oplus H_n(A'') \xrightarrow{j'_* - j''_*} H_n(X) \\ \xrightarrow{\partial} & H_{n-1}(A' \cap A'') \to \cdots \end{aligned}$$

of abelian groups.

Exercise 6. Let X be a chain complex and suppose $X \subseteq CX$ is a subcomplex of some CX satisfying $H_*(CX) = 0$. Define ΣX by the pushout diagram of inclusions

$$(4) \qquad \begin{array}{ccc} X & \longrightarrow & CX \\ \big\downarrow & & \big\downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

in **Ch**. In other words, define $\Sigma X := CX \cup_X CX$. By analogy with topological spaces, we are thinking of CX as the "cone on X" and ΣX as the "suspension of X". Let $n \in \mathbb{Z}$ be any integer. Prove the following.

(a) $H_n(X) \cong H_{n+1}(\Sigma X)$,

The idea is to use Proposition 2.

We would like to construct a chain complex CX which satisfies the assumptions in Exercise 6. Given any chain complex X, the *cone on* X, denoted by CX, and the corresponding inclusion $i : X \longrightarrow CX$ map, are defined by

$$\begin{aligned} (CX)_n &:= X_{n-1} \oplus X_n, & \partial(x, y) &:= (-\partial x, x + \partial y), \\ i : X &\longrightarrow CX, & y &\longmapsto (0, y). \end{aligned}$$

Exercise 7. Let X be a chain complex. Prove the following.

(a) The differential ∂ of CX satisfies $\partial\partial = 0$.
(b) The map $i : X \longrightarrow CX$ is a well-defined map of chain complexes.
(c) $H_*(CX) = 0$.

There is a simpler (alternative) construction for a chain complex ΣX which satisfies the isomorphisms in Exercise 6(a). The idea is to replace one of the CX in (4) with the trivial chain complex 0; note that $H_*(CX) = 0 = H_*(0)$.

Given any chain complex X, the *suspension of* X, denoted by ΣX, is defined by the pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

in **Ch**. In other words, $\Sigma X = CX/X$ and hence fits into the short exact sequence

$$0 \to X \xrightarrow{i} CX \to \Sigma X \to 0. \tag{5}$$

Exercise 8. Let $n \in \mathbb{Z}$ be any integer. Prove the following.

(a) $(\Sigma X)_n = X_{n-1}$.
(b) $\partial_n|_{\Sigma X} = -\partial_{n-1}|_X$.
(c) $H_n(X) \cong H_{n+1}(\Sigma X)$.

Remark 3. An isomorphic definition of ΣX, which eliminates the sign change on the differential in Exercise 8(b), is given by

$$(\Sigma X)_n := X_{n-1}, \qquad \partial_n|_{\Sigma X} := \partial_{n-1}|_X.$$

Since this definition of ΣX is isomorphic to CX/X, it follows immediately that this ΣX fits into a short exact sequence of the form (5).

Exercise 9. Consider any commutative diagram of the form

$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \big\downarrow & \text{I} & \big\downarrow & \text{II} & \big\downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}$$

in **Ch**. Prove the following.

(a) If I and II are pushout diagrams, then the outer diagram (with top and bottom edges the evident composites) is a pushout diagram.
(b) If I and the outer diagram are pushout diagrams, then II is a pushout diagram.

Exercise 10. Let X be a chain complex and suppose $A \subseteq X$ is a subcomplex. Consider any pushout diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle i}\big\downarrow{\scriptstyle \subseteq} & & \big\downarrow{\scriptstyle j} \\ X & \longrightarrow & X \oplus_A B \end{array}$$

in **Ch**. Prove the following.

(a) j is a monomorphism.
(b) $(X \oplus_A B)/B \cong X/A$. In other words, $\operatorname{coker}(j) \cong X/A$.
(c) If i is a homology isomorphism, then j is a homology isomorphism.

For (a), try using the construction in Remark 1. For (b), consider Exercise 9(a). For (c), consider the short exact sequence

$$0 \to A \xrightarrow{i} X \to X/A \to 0$$

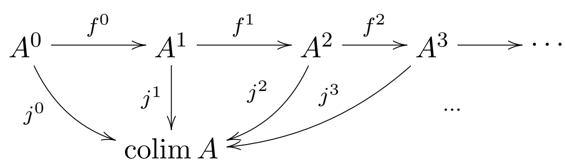
of chain complexes, and note (by the Five Lemma) that i is a homology isomorphism if and only if $H_*(X/A) = 0$.

Denote by $\mathbf{I}$ the category associated to the set $\{0, 1, 2, \cdots\}$ equipped with its natural ordering. It is easy to check that giving a functor $A : \mathbf{I} \longrightarrow \mathbf{Ch}$, is the same as giving a diagram of the form

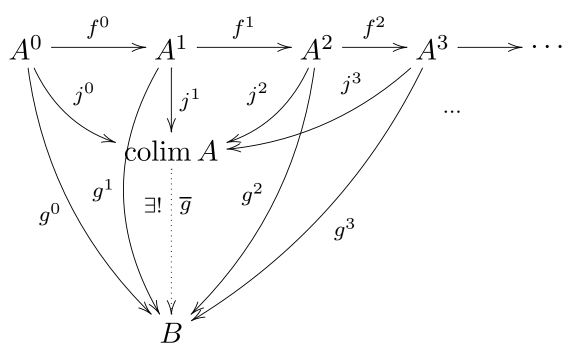
$$A^0 \xrightarrow{f^0} A^1 \xrightarrow{f^1} A^2 \xrightarrow{f^2} A^3 \longrightarrow \cdots$$

in $\mathbf{Ch}$. For this reason, such a functor A is sometimes called an $\mathbf{I}$-*shaped diagram* in chain complexes.

It will be useful to generalize the union of a sequence of subcomplexes. Suppose A is an $\mathbf{I}$-shaped diagram in chain complexes. The *colimit* of the diagram A, written $\operatorname{colim} A$ (or $\operatorname{colim}_k A^k$ or $\operatorname{colim}_{\mathbf{I}} A$ or $\varinjlim A$), is a chain complex, together with the commutative diagram

(6) 

of chain complexes, which satisfies the universal property: given a collection of maps $\{A^k \xrightarrow{g^k} B\}_{k \geq 0}$ of chain complexes such that $g^{k+1} f^k = g^k$, for every $k \geq 0$, then there exists a unique map $\overline{g}$ of chain complexes such that the diagram



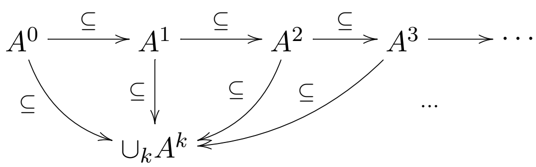
in $\mathbf{Ch}$ commutes (i.e., such that $\overline{g} j^k = g^k$ for every $k \geq 0$).

Remark 4. It is easy to verify that such colimits in chain complexes exist; define $\operatorname{colim} A := (\oplus_{k \geq 0} A^k)/\sim$, the direct sum of the chain complexes A^k modulo $f^k(a^k) \sim a^k$, $a^k \in A^k$, $k \geq 0$.

Exercise 11. Let X be a chain complex and suppose $\{A^k\}_{k\geq 0}$ is a collection of subcomplexes of X such that

$$A^0 \subseteq A^1 \subseteq A^2 \subseteq \cdots \subseteq \cup_k A^k = X.$$

(a) Prove that the diagram of inclusions



in **Ch** satisfies the universal property of colimits. Conclude that $\operatorname{colim}_k A^k \cong \cup_k A^k$.

Exercise 12. Consider any **I**-shaped diagram

$$A^0 \xrightarrow{f^0} A^1 \xrightarrow{f^1} A^2 \xrightarrow{f^2} A^3 \longrightarrow \cdots$$

in **Ch**. Let $n \in \mathbb{Z}$ be any integer and note that applying $H_n(-)$ gives an **I**-shaped diagram in abelian groups. Prove the following.

(a) The induced map $\operatorname{colim}_k H_n(A^k) \xrightarrow{\cong} H_n(\operatorname{colim}_k A^k)$ is an isomorphism.

In other words, homology $H_*(-)$ commutes with sequential colimits. This is proved in [3, Theorem 4.1.7].

Exercise 13. Let $g\colon A \longrightarrow B$ be a morphism of **I**-shaped diagrams

$$\begin{array}{cccccccccc}
A: & & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & A^3 & \longrightarrow \cdots \\
\downarrow g & & \downarrow g^0 & & \downarrow g^1 & & \downarrow g^2 & & \downarrow g^3 & \\
B: & & B^0 & \longrightarrow & B^1 & \longrightarrow & B^2 & \longrightarrow & B^3 & \longrightarrow \cdots
\end{array}$$

in **Ch**. Prove the following.

(a) If g is an objectwise homology isomorphism, then the induced map $\operatorname{colim}_k A^k \longrightarrow \operatorname{colim}_k B^k$ is a homology isomorphism.

In other words, $\operatorname{colim}_{\mathbf{I}}(-)$ respects homology isomorphisms. The idea is to use Exercise 12.

Here are some references for this material: [1, Chapter II], [2, Chapter III.3], [3, Chapters 0.1 and 4].

References

[1] S. Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.

[2] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[3] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York, 1981. Corrected reprint.