

in **Set** such that the simplicial identities are satisfied, and a map of simplicial sets $f : X \rightarrow Y$ is a sequence of maps $f_n : X_n \rightarrow Y_n$ ($n \geq 0$) which respect the face and degeneracy maps.

Recall that a category \mathbf{D} is *small* if the collection $\text{Ob}(\mathbf{D})$ of objects is a set; note that we have already assumed in the definition of category that the collection

$$\mathbf{D}(a, b) = \{f \mid f : a \rightarrow b\}$$

of all morphisms in \mathbf{D} from a to b forms a set, for every pair $a, b \in \mathbf{D}$.

The following is an important tool for constructing certain simplicial sets from small categories. In fact, simplicial sets can be thought of as an abstraction of the nerve of a small category \mathbf{D} ; i.e., the simplicial identities (Definition 1) can be reconstructed simply by examining all naturally occurring identities satisfied by $N(\mathbf{D})$.

Definition 2. Let \mathbf{D} be a small category. The *nerve of \mathbf{D}* , denoted by $N(\mathbf{D})$, is the simplicial set with n -simplices the length n sequences

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n$$

of composable morphisms in \mathbf{D} ; the 0-simplices are the objects in \mathbf{D} . The face maps $d_i : N(\mathbf{D})_n \rightarrow N(\mathbf{D})_{n-1}$ are defined by “deleting the object a_i ”; i.e., for each $0 < i < n$ we have

$$\begin{aligned} d_0 : a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n &\longmapsto a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n \\ d_n : a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n &\longmapsto a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} a_{n-1} \\ d_i : a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n &\longmapsto \\ &a_0 \rightarrow \cdots \rightarrow a_{i-1} \xrightarrow{\alpha_{i+1}\alpha_i} a_{i+1} \rightarrow \cdots \rightarrow a_n \end{aligned}$$

The degeneracy maps $s_j : N(\mathbf{D})_n \rightarrow N(\mathbf{D})_{n+1}$ are defined by “repeating the object a_j ”; i.e., we have

$$\begin{aligned} s_j : a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n &\longmapsto \\ &a_0 \rightarrow \cdots \rightarrow a_{j-1} \xrightarrow{\alpha_j} a_j \xrightarrow{\text{id}} a_j \xrightarrow{\alpha_{j+1}} a_{j+1} \rightarrow \cdots \rightarrow a_n \end{aligned}$$

The purpose of Exercise 1 is to gain some familiarity with the simplicial identities (Definition 1), and to gain experience with the nerve of a small category.

Exercise 1. Let \mathbf{D} be a small category and consider the face and degeneracy maps of the nerve $N(\mathbf{D})$ of \mathbf{D}

$$\begin{array}{ccccccc} & & & & \rightrightarrows & & \\ & & & & \rightrightarrows & & \\ & & & & \rightrightarrows & & \\ \xrightarrow{s_0} & & \rightrightarrows & & \rightrightarrows & & \\ \xrightarrow{d_0} & N(\mathbf{D})_0 & \xleftarrow{d_1} & N(\mathbf{D})_1 & \xleftarrow{d_2} & N(\mathbf{D})_2 & \xleftarrow{d_3} & N(\mathbf{D})_3 \cdots \end{array}$$

as indicated in Definition 2. For warm-up, consider the corresponding truncated diagram

$$(1) \quad \begin{array}{ccccc} & & \xrightarrow{s_0} & & \rightrightarrows \\ & & \downarrow d_0 & & \\ N(\mathbf{D})_0 & \xleftarrow{d_0} & N(\mathbf{D})_1 & \xleftarrow{d_0} & N(\mathbf{D})_2 \\ & \downarrow d_1 & & \downarrow d_1 & \\ & & & & \end{array}$$

(a) Check that the following

$$\begin{array}{llllll} d_0 d_1 = d_0 d_0 & & & & d_0 s_0 = \text{id} & \\ d_0 d_2 = d_1 d_0 & s_0 s_0 = s_1 s_0 & d_0 s_1 = s_0 d_0 & & d_1 s_1 = \text{id} & d_2 s_0 = s_0 d_1 \\ d_1 d_2 = d_1 d_1 & & & & d_1 s_0 = \text{id} & \\ & & & & d_2 s_1 = \text{id} & \end{array}$$

is a complete list of the simplicial identities which can be written in terms of the face and degeneracy maps appearing in diagram (1).

- (b) Verify that the face and degeneracy maps in (1) satisfy the simplicial identities listed in (a).
(c) More generally, prove that $N(\mathbf{D})$ is a simplicial set; i.e., verify that $N(\mathbf{D})$ satisfies the simplicial identities in Definition 1.

Denote by **Cat** the category of small categories and their functors.

Exercise 2. Prove the following:

(a) The nerve construction

$$N : \mathbf{Cat} \longrightarrow \mathbf{sSet}, \quad \mathbf{D} \longmapsto N(\mathbf{D}),$$

gives a well-defined functor; i.e., verify that $N(gf) = N(g)N(f)$ and $N(\text{id}) = \text{id}$.

(b) The composite functor

$$\mathbf{Set} \rightarrow \mathbf{Cat} \xrightarrow{N} \mathbf{sSet}$$

associates to each set T the constant simplicial set with value T ; i.e., $N(T)_n = T$ for each $n \geq 0$.

The purpose of the following is to prove that simplicial sets are the same as Δ^{op} -shaped diagrams in **Set**.

Recall from lecture the following definitions; please note that the maps δ^i and σ^j introduced in lecture are here denoted by d^i and s^j , respectively.

Definition 3. Define the ordered sets $[n] := \{0, 1, \dots, n\}$ for each $n \geq 0$, equipped with their natural ordering.

- Δ is the category with objects the ordered sets $[n]$ for $n \geq 0$ and morphisms the non-decreasing maps $\xi : [n] \rightarrow [n']$; i.e., such that $k \leq l$ implies $\xi(k) \leq \xi(l)$.
- For each $n \geq 0$, the *standard n -simplex* $\Delta[n]$ is the simplicial set with k -simplices the morphisms in Δ from $[k]$ to $[n]$; i.e., $\Delta[n]_k := \Delta([k], [n]) = \text{hom}_{\Delta}([k], [n])$.

- For each $n \geq 1$, the map $d^i : [n-1] \rightarrow [n]$ in Δ is the increasing injection which does not take the value $i \in [n]$; here, $(0 \leq i \leq n)$.
- For each $n \geq 0$, the map $s^j : [n+1] \rightarrow [n]$ in Δ is the non-decreasing surjection which takes twice the value $j \in [n]$; here, $(0 \leq j \leq n)$.

Proposition 4. *The following identities*

$$\begin{aligned}
 d^j d^i &= d^i d^{j-1}, & i < j, \\
 s^j s^i &= s^i s^{j+1}, & i \leq j, \\
 s^j d^i &= \begin{cases} d^i s^{j-1}, & i < j, \\ \text{id}, & i = j, \\ d^{i-1} s^j, & i > j + 1 \end{cases} & i = j + 1
 \end{aligned}$$

are satisfied by the maps d^i and s^j in Δ (Definition 3).

We will prove this proposition in Exercise 3. Observe that the identities for d^i and s^j in Proposition 4 are “dual” to the simplicial identities for d_i and s_j in Definition 1. For this reason, the identities in Proposition 4 are sometimes called the *cosimplicial identities*.

Exercise 3. Consider the maps d^i and s^j in Δ which we picture here by the following diagram

$$\begin{array}{ccccccc}
 & \xleftarrow{s^0} & & \xleftarrow{\quad} & \xleftarrow{\quad} & \xleftarrow{\quad} & \\
 & \xleftarrow{d^0} & & \xleftarrow{\quad} & \xleftarrow{\quad} & \xleftarrow{\quad} & \\
 [0] & \xrightarrow{d^0} & [1] & \xrightarrow{\quad} & [2] & \xrightarrow{\quad} & [3] \cdots \\
 & \xrightarrow{d^1} & & \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} &
 \end{array}$$

in **Set**. For warm-up, consider the corresponding truncated diagram

$$(2) \quad \begin{array}{ccc}
 & \xleftarrow{s^0} & \xleftarrow{\quad} \\
 & \xleftarrow{d^0} & \xleftarrow{\quad} \\
 [0] & \xrightarrow{d^0} & [1] \xrightarrow{\quad} [2] \\
 & \xrightarrow{d^1} &
 \end{array}$$

(a) Check that the following

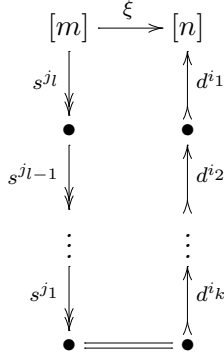
$$\begin{array}{llllll}
 d^1 d^0 = d^0 d^0 & & & & s^0 d^0 = \text{id} & \\
 d^2 d^0 = d^0 d^1 & s^0 s^0 = s^0 s^1 & s^1 d^0 = d^0 s^0 & & s^1 d^1 = \text{id} & s^0 d^2 = d^1 s^0 \\
 d^2 d^1 = d^1 d^1 & & & & s^0 d^1 = \text{id} & \\
 & & & & s^1 d^2 = \text{id} &
 \end{array}$$

is a complete list of the identities in Proposition 4 which can be written in terms of the maps d^i and s^j appearing in (2).

- (b) Verify that the maps d^i and s^j in (2) satisfy the identities in (a).
(c) More generally, prove Proposition 4.

It turns out that the maps d^i and s^j together with the relations in Proposition 4 generate the category Δ . The proposition is this.

Proposition 5. Every map $\xi : [m] \rightarrow [n]$ in Δ has a unique factorization of the form $\xi = d^{i_1} \dots d^{i_k} s^{j_1} \dots s^{j_l}$ satisfying



$n \geq i_1 > \dots > i_k \geq 0$, $0 \leq j_1 < \dots < j_l < m$, and $n - k + l = m$.

Proof. For existence, choose the indexing sets as follows:

$$\begin{aligned}
 \{i_*\} &:= \{i \in [n] \mid i \notin \text{image}(\xi)\}, \\
 \{j_*\} &:= \{j \in [m] \mid \xi(j) = \xi(j+1)\}.
 \end{aligned}$$

For uniqueness, note that existence of a factorization satisfying the indicated conditions implies this choice of indexing sets. \square

The purpose of Exercise 4 is to prove that a simplicial set X is the same as a Δ^{op} -shaped diagram in **Set**; i.e., a functor $X : \Delta^{\text{op}} \rightarrow \mathbf{Set}$.

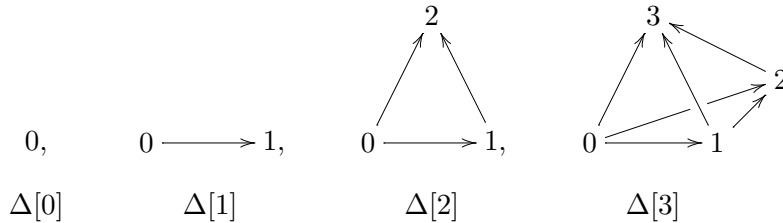
Exercise 4.

- (a) Verify the details in the proof of Proposition 5.
- (b) Prove that $\mathbf{sSet} = \mathbf{Set}^{\Delta^{\text{op}}}$

Definition 6. Let X be a simplicial set. An n -simplex $x \in X_n$ is *degenerate* if $x = s_j x'$ for some j and some $x' \in X_{n-1}$. Every vertex is non-degenerate.

Exercise 5. Let $n \geq 0$. Prove the following.

- (a) $\Delta[n] \cong \mathbf{N}([n])$; here, on the right-hand side, we denote by $[n]$ the category associated to the set $[n] := \{0, 1, \dots, n\}$, equipped with its natural ordering (Series 2, Exercise 1).
- (b) Find all of the non-degenerate n -simplices ($n=2,3,4$) in the simplicial sets $\Delta[0], \Delta[1], \Delta[2]$, and $\Delta[3]$. The following pictures



suggest the result.

To get started on (b), try using (a) and the simple characterization of degenerate n -simplices in the nerve of a category.

A short proof of the following can be found in [2, II.3].

Proposition 7. *Let X be a simplicial set. For each n -simplex x of X , there is a surjection $s : [n] \rightarrow [k]$ in Δ and a non-degenerate k -simplex y such that $x = X(s)(y)$. Furthermore, the pair (s, y) is unique.*

A consequence is that every simplicial set has a canonical decomposition. The proposition is this.

Proposition 8. *Let X be a simplicial set. For each $k \geq 0$, denote by $(\mathcal{N}X)_k$ the set of non-degenerate k -simplices of X . There is an isomorphism between X and a simplicial set of the form*

$$(\mathcal{N}X)_0 \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} (\mathcal{N}X)_0 \amalg (\mathcal{N}X)_1 \begin{array}{c} \xrightarrow{\quad} \\ \longleftarrow \end{array} (\mathcal{N}X)_0 \amalg (\mathcal{N}X)_1 \amalg (\mathcal{N}X)_1 \amalg (\mathcal{N}X)_2 \cdots$$

which is given objectwise by isomorphisms

$$\coprod_{\substack{[n] \rightarrow [k] \\ \text{in } \Delta}} (\mathcal{N}X)_k \xrightarrow{\cong} X_n$$

Here, the indicated coproduct is indexed over the set of all surjections in Δ from $[n]$ to $[k]$.

Exercise 6. The purpose of this exercise is to gain some experience with the decomposition in Proposition 8.

- (a) Find a naturally occurring map of the form

$$\coprod_{\substack{[n] \rightarrow [k] \\ \text{in } \Delta}} (\mathcal{N}X)_k \longrightarrow X_n.$$

- (b) Find naturally occurring face and degeneracy maps for the

$$\Delta^{\text{op}} \longrightarrow \mathbf{Set}, \quad [n] \longmapsto \coprod_{\substack{[n] \rightarrow [k] \\ \text{in } \Delta}} (\mathcal{N}X)_k.$$

indicated simplicial set.

Exercise 7. Let X be a simplicial set and $n \geq 0$. Prove the following.

- If $X_n = *$, then $X_k = *$ for every $0 \leq k \leq n$.
- If $X_n = \emptyset$, then $X_k = \emptyset$ for every $k \geq 0$.
- There exist simplicial sets Y such that $Y \not\cong N(\mathbf{D})$ for every small category \mathbf{D} ; i.e., not every simplicial set is the nerve of a small category.

Consider using the decomposition in Proposition 8.

Here are some references for this material: [1], [2, II.3], [3, VIII.5].

REFERENCES

- [1] W. G. Dwyer and H. Henn. *Homotopy theoretic methods in group cohomology*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2001.
- [2] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.
- [3] S. Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.