Homology and cohomology **Spring 2009**
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Series 6

To work effectively with simplicial sets, which provide useful algebraic or combinatorial models of topological spaces, we need to get serious about geometric realization, colimits, limits, and adjunctions. A first step is to recall the Yoneda Lemma. Given any pair of functors $F, G : \mathbf{D} \longrightarrow \mathbf{C}$, denote by

$$\mathrm{Nat}(F, G) := \{\varphi \mid \varphi : F \longrightarrow G\}$$

the collection of all natural transformations from F to G. Denote by **sSet** the category of simplicial sets and their maps, and recall from Series 5 that $\mathbf{sSet} = \mathbf{Set}^{\Delta^{\mathrm{op}}}$. Hence, a simplicial set Y is the same as a Δ^{op}-shaped diagram $Y : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$, and $\mathbf{sSet}(X, Y) = \mathrm{Nat}(X, Y)$.

Proposition 1 (The Yoneda Lemma)**.** *Let* $\mathbf{D}$ *be a category and consider any* $r, s \in \mathbf{D}$*.*

(a) *If* $X : \mathbf{D} \longrightarrow \mathbf{Set}$ *is a functor, then there exists an isomorphism*

$$\mathrm{Nat}\big(\mathbf{D}(r, -), X\big) \cong X(r)$$

natural in r, X*.*

(b) *If* $Y : \mathbf{D}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ *is a functor, then there exists an isomorphism*

$$\mathrm{Nat}\big(\mathbf{D}(-, s), Y\big) \cong Y(s)$$

natural in s, Y*.*

Part (a) of Proposition 1 is proved below, and part (b) is Exercise 1.

Exercise 1. Complete the proof of Proposition 1.

Proof of Proposition 1. Consider part (a). Let $\varphi : \mathbf{D}(r, -) \longrightarrow X$ be a natural transformation from $\mathbf{D}(r, -)$ to X. Then for every morphism $f : s \to s'$ in $\mathbf{D}$ the diagram

$$\begin{array}{ccccc} s & \qquad & \mathbf{D}(r, s) & \xrightarrow{\varphi_s} & X(s) \\ {\scriptstyle f}\downarrow & & {\scriptstyle (\mathrm{id}, f)}\downarrow & & \downarrow{\scriptstyle X(f)} \\ s' & & \mathbf{D}(r, s') & \xrightarrow{\varphi_{s'}} & X(s') \end{array}$$

in **Set** commutes. In particular, for every $f : r \to s$ in $\mathbf{D}$ the diagram

$$\begin{array}{ccccc} r & \qquad & \mathbf{D}(r, r) & \xrightarrow{\varphi_r} & X(r) \\ {\scriptstyle f}\downarrow & & {\scriptstyle (\mathrm{id}, f)}\downarrow & & \downarrow{\scriptstyle X(f)} \\ s & & \mathbf{D}(r, s) & \xrightarrow{\varphi_s} & X(s) \end{array}$$

commutes. Chasing the identity map $\mathrm{id} : r \to r$ around the diagram verifies

$$\begin{array}{ccccc} (r \xrightarrow{\mathrm{id}} r) & \longmapsto & \varphi_r(\mathrm{id}) & \longmapsto & X(f)\varphi_r(\mathrm{id}) \\ (r \xrightarrow{\mathrm{id}} r) & \longmapsto & (r \xrightarrow{\mathrm{id}} r \xrightarrow{f} s) & \longmapsto & \varphi_s(f) \end{array}$$

that $\varphi_s(f) = X(f)\varphi_r(\mathrm{id})$. In other words, φ is completely determined by its value on the identity map $\mathrm{id} : r \to r$. It is easy to check that the map

$$\mathrm{Nat}\big(\mathbf{D}(r, -), X\big) \xrightarrow{\cong} X(r), \qquad \varphi \longmapsto \varphi_r(\mathrm{id}),$$

is an isomorphism, natural in r, X. The proof of part (b) is similar. □

Exercise 2. Use the Yoneda Lemma to prove the following.

(a) If Y is a simplicial set, then there is an isomorphism

$$\mathbf{sSet}(\Delta[n], Y) \cong Y_n$$

natural in n, Y.

(b) If $f : \Delta[n] \longrightarrow K$ is a map of simplicial sets, then f is completely determined by $f(0, \cdots, n) \in K_n$. Here, $(0, \cdots, n)$ denotes the unique non-degenerate n-simplex of $\Delta[n]$.

In other words, giving an n-simplex $y \in Y_n$ is the same as giving a map $\Delta[n] \xrightarrow{y} Y$ of simplicial sets.

Exercise 3. Let $\mathbf{D}$ be a category. Prove the following.

(a) For each $r, r' \in \mathbf{D}$, there are natural isomorphisms

$$\mathrm{Nat}\big(\mathbf{D}(r, -), \mathbf{D}(r', -)\big) \cong \mathbf{D}(r', r).$$

(b) For each $s, s' \in \mathbf{D}$, there are natural isomorphisms

$$\mathrm{Nat}\big(\mathbf{D}(-, s), \mathbf{D}(-, s')\big) \cong \mathbf{D}(s, s').$$

(c) Conclude that for each $m, n \geq 0$, there are natural isomorphisms

$$\mathbf{sSet}(\Delta[m], \Delta[n]) \cong \Delta([m], [n]).$$

In other words, giving a map $\xi_* : \Delta[m] \longrightarrow \Delta[n]$ of simplicial sets is the same as giving a map $\xi : [m] \longrightarrow [n]$ in Δ.

Recall from lecture the following definitions.

Definition 2. Let X be a simplicial set and Y a topological space.

- For each $n \geq 0$, the *standard topological n-simplex* Δ^n is the subspace defined by

$$\Delta^n := \Big\{(t_0, \cdots, t_n) \,\big|\, 0 \leq t_i \leq 1, \ \sum_{i=0}^{n} t_i = 1\Big\} \subseteq \mathbb{R}^{n+1}.$$

Define $d^i : \Delta^{n-1} \longrightarrow \Delta^n$ and $s^j : \Delta^{n+1} \longrightarrow \Delta^n$ for $0 \leq i, j \leq n$ by

$$d^i(t_0, \cdots, t_{n-1}) = (t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1}),$$
$$s^j(t_0, \cdots, t_{n+1}) = (t_0, \cdots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \cdots, t_{n+1}).$$

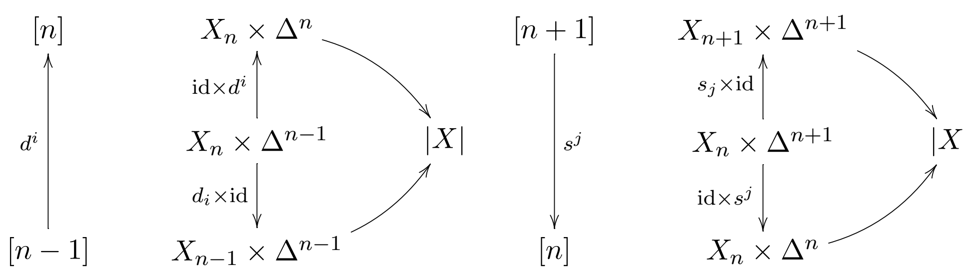
- For each $n \geq 0$, the *singular complex* of Y, denoted by $S_\bullet(Y)$, is the simplicial set with n-simplices defined by

$$S_n(Y) := \mathbf{Top}(\Delta^n, Y).$$

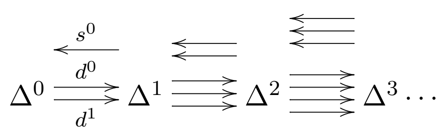
- The *geometric realization* of X, denoted by $|X|$, is the topological space defined by

$$|X| = \Big(\coprod_{n\geq 0} X_n \times \Delta^n\Big)/\sim$$

modulo $(d_i x, t) \sim (x, d^i t)$ and $(s_j x, t) \sim (x, s^j t)$, which may be pictured by the diagrams



It is easy to check that the cosimplicial identities ("dual" to the simplicial identities) are satisfied by the d^i and s^j maps which we picture here by the diagram



in **Top**. Hence, the topological n-simplices Δ^n ($n \geq 0$) determine a functor

$$\Delta^\bullet : \Delta \longrightarrow \mathbf{Top}, \qquad [n] \longmapsto \Delta^n.$$

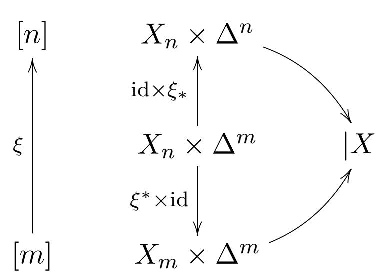
This will be useful for the following exercise.

Exercise 4. Let X be a simplicial set.

(a) Prove that geometric realization $|X|$ satisfies

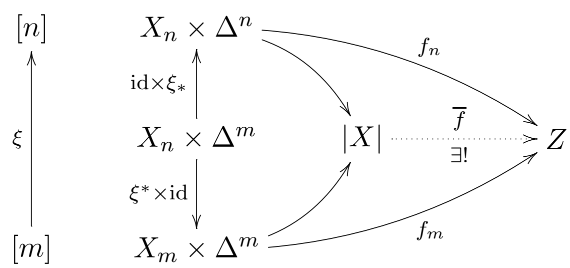
$$|X| = \Big(\coprod_{n\geq 0} X_n \times \Delta^n\Big)/\sim$$

modulo $(\xi^* x, t) \sim (x, \xi_* t)$, which may be pictured by the diagrams



(b) Conclude that $|X|$ satisfies the universal property: given any collection of maps $\{f_n : X_n \times \Delta^n \longrightarrow Z\}_{n\geq 0}$ such that the outer diagram in (1) commutes, then there exists a unique continuous map $\overline{f}$ such that the diagram

(1)



in **Top** commutes for all maps ξ in Δ.

(c) Prove that $|\Delta[k]| \cong \Delta^k$.

For part (c), try looking for naturally occurring maps $\{\Delta[k]_n \times \Delta^n \longrightarrow \Delta^k\}_{n\geq 0}$ and verify the universal property of geometric realization.

Remark 3. Recall from lecture that geometric realization

$$|-| : \mathbf{sSet} \longrightarrow \mathbf{Top}, \quad X \longmapsto |X|$$

gives a well-defined functor. This can also be deduced directly from the universal property.

Proposition 4. *Let X be a simplicial set and Y a topological space. There are isomorphisms*

$$\mathbf{Top}(|X|, Y) \cong \mathbf{sSet}(X, S_\bullet(Y))$$

natural in X, Y. In other words, there is an adjunction

$$\mathbf{sSet} \underset{S_\bullet(-)}{\overset{|-|}{\rightleftarrows}} \mathbf{Top}$$

with left adjoint the geometric realization functor $|-|$ and right adjoint the singular complex functor $S_\bullet(-)$.

Exercise 5. Prove Proposition 4.

To get started on Exercise 5, consider any map $f : X \longrightarrow S_\bullet(Y)$ of simplicial sets, then this is the same as giving a collection of maps $f_n : X_n \longrightarrow S_n(Y)$ ($n \geq 0$) in **Set** such that certain diagrams commute, which is the same as giving a corresponding collection of maps $f_n : X_n \times \Delta^n \longrightarrow Y$ ($n \geq 0$) in **Top** such that certain diagrams commute.

In algebraic topology and homotopy theory, the existence of mapping spaces between any pair Y, Z of spaces is frequently useful; for example, loop spaces, path spaces, and homotopy limits are often defined or constructed using a topological space of maps. Similarly, when working with simplicial sets, it is frequently useful to have available a simplicial set of maps between any pair Y, Z of simplicial sets.

Definition 5. Let Y, Z be simplicial sets. The *simplicial function complex*, denoted by $\mathrm{Map}(Y, Z)$, is the simplicial set with n-simplices defined by

$$\mathrm{Map}(Y, Z)_n := \mathbf{sSet}(Y \times \Delta[n], Z).$$

The following proposition is Exercise 6(b).

Proposition 6. *Let X, Y, Z be simplicial sets. There are isomorphisms*

$$\mathbf{sSet}(X \times Y, Z) \cong \mathbf{sSet}(X, \mathrm{Map}(Y, Z))$$

natural in X, Y, Z. In particular, there is an adjunction

$$\mathbf{sSet} \underset{\mathrm{Map}(Y,-)}{\overset{-\times Y}{\rightleftarrows}} \mathbf{sSet}$$

with left adjoint the functor $- \times Y$ and right adjoint the functor $\mathrm{Map}(Y, -)$.

Exercise 6. Let X, Y, Z be simplicial sets. Prove the following.

(a) Use Exercise 3(c) to define naturally occurring face and degeneracy maps for $\mathrm{Map}(X, Y)$ and verify these satisfy the simplicial identities; it is equivalent to use Exercise 3(c) to verify that $\mathrm{Map}(X, Y)$ is a well-defined functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$.

(b) There are isomorphisms

$$\mathbf{sSet}(X \times Y, Z) \cong \mathbf{sSet}(X, \mathrm{Map}(Y, Z))$$

natural in X,Y,Z.

(c) There are isomorphisms

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

natural in X,Y,Z.

(d) $X \cong \mathrm{Map}(*, X)$; here, $* = \Delta[0]$.

Parts (c) and (d) follow easily from (b) and the Yoneda Lemma (see, Exercise 3). For example, for (c) it suffices to verify that

$$\mathbf{sSet}(-, \mathrm{Map}(X \times Y, Z)) \cong \mathbf{sSet}(-, \mathrm{Map}(X, \mathrm{Map}(Y, Z))).$$

Hence, it is enough to verify that there are natural isomorphisms

$$\mathbf{sSet}(A, \mathrm{Map}(X \times Y, Z)) \cong \mathbf{sSet}(A, \mathrm{Map}(X, \mathrm{Map}(Y, Z))).$$

Here are some references for this material: [1, Section 3], [2, Chapter III], [3, Chapter 16].

References

[1] W. G. Dwyer and H. Henn. *Homotopy theoretic methods in group cohomology*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2001.

[2] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[3] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. Available at: `http://www.math.uchicago.edu/ may/` .