

Series 6

To work effectively with simplicial sets, which provide useful algebraic or combinatorial models of topological spaces, we need to get serious about geometric realization, colimits, limits, and adjunctions. A first step is to recall the Yoneda Lemma. Given any pair of functors  $F, G : \mathbf{D} \rightarrow \mathbf{C}$ , denote by

$$\text{Nat}(F, G) := \{\varphi \mid \varphi : F \rightarrow G\}$$

the collection of all natural transformations from  $F$  to  $G$ . Denote by  $\mathbf{sSet}$  the category of simplicial sets and their maps, and recall from Series 5 that  $\mathbf{sSet} = \mathbf{Set}^{\Delta^{\text{op}}}$ . Hence, a simplicial set  $Y$  is the same as a  $\Delta^{\text{op}}$ -shaped diagram  $Y : \Delta^{\text{op}} \rightarrow \mathbf{Set}$ , and  $\mathbf{sSet}(X, Y) = \text{Nat}(X, Y)$ .

**Proposition 1** (The Yoneda Lemma). *Let  $\mathbf{D}$  be a category and consider any  $r, s \in \mathbf{D}$ .*

(a) *If  $X : \mathbf{D} \rightarrow \mathbf{Set}$  is a functor, then there exists an isomorphism*

$$\text{Nat}(\mathbf{D}(r, -), X) \cong X(r)$$

*natural in  $r, X$ .*

(b) *If  $Y : \mathbf{D}^{\text{op}} \rightarrow \mathbf{Set}$  is a functor, then there exists an isomorphism*

$$\text{Nat}(\mathbf{D}(-, s), Y) \cong Y(s)$$

*natural in  $s, Y$ .*

Part (a) of Proposition 1 is proved below, and part (b) is Exercise 1.

**Exercise 1.** Complete the proof of Proposition 1.

*Proof of Proposition 1.* Consider part (a). Let  $\varphi : \mathbf{D}(r, -) \rightarrow X$  be a natural transformation from  $\mathbf{D}(r, -)$  to  $X$ . Then for every morphism  $f : s \rightarrow s'$  in  $\mathbf{D}$  the diagram

$$\begin{array}{ccc} s & \mathbf{D}(r, s) & \xrightarrow{\varphi_s} X(s) \\ f \downarrow & (\text{id}, f) \downarrow & \downarrow X(f) \\ s' & \mathbf{D}(r, s') & \xrightarrow{\varphi_{s'}} X(s') \end{array}$$

in  $\mathbf{Set}$  commutes. In particular, for every  $f : r \rightarrow s$  in  $\mathbf{D}$  the diagram

$$\begin{array}{ccc} r & \mathbf{D}(r, r) & \xrightarrow{\varphi_r} X(r) \\ f \downarrow & (\text{id}, f) \downarrow & \downarrow X(f) \\ s & \mathbf{D}(r, s) & \xrightarrow{\varphi_s} X(s) \end{array}$$

commutes. Chasing the identity map  $\text{id} : r \rightarrow r$  around the diagram verifies

$$\begin{aligned} (r \xrightarrow{\text{id}} r) &\longmapsto \varphi_r(\text{id}) \longmapsto X(f)\varphi_r(\text{id}) \\ (r \xrightarrow{\text{id}} r) &\longmapsto (r \xrightarrow{\text{id}} r \xrightarrow{f} s) \longmapsto \varphi_s(f) \end{aligned}$$

that  $\varphi_s(f) = X(f)\varphi_r(\text{id})$ . In other words,  $\varphi$  is completely determined by its value on the identity map  $\text{id} : r \rightarrow r$ . It is easy to check that the map

$$\text{Nat}(\mathbf{D}(r, -), X) \xrightarrow{\cong} X(r), \quad \varphi \longmapsto \varphi_r(\text{id}),$$

is an isomorphism, natural in  $r, X$ . The proof of part (b) is similar.  $\square$

**Exercise 2.** Use the Yoneda Lemma to prove the following.

- (a) If  $Y$  is a simplicial set, then there is an isomorphism

$$\mathbf{sSet}(\Delta[n], Y) \cong Y_n$$

natural in  $n, Y$ .

- (b) If  $f : \Delta[n] \rightarrow K$  is a map of simplicial sets, then  $f$  is completely determined by  $f(0, \dots, n) \in K_n$ . Here,  $(0, \dots, n)$  denotes the unique non-degenerate  $n$ -simplex of  $\Delta[n]$ .

In other words, giving an  $n$ -simplex  $y \in Y_n$  is the same as giving a map  $\Delta[n] \xrightarrow{y} Y$  of simplicial sets.

**Exercise 3.** Let  $\mathbf{D}$  be a category. Prove the following.

- (a) For each  $r, r' \in \mathbf{D}$ , there are natural isomorphisms

$$\text{Nat}(\mathbf{D}(r, -), \mathbf{D}(r', -)) \cong \mathbf{D}(r', r).$$

- (b) For each  $s, s' \in \mathbf{D}$ , there are natural isomorphisms

$$\text{Nat}(\mathbf{D}(-, s), \mathbf{D}(-, s')) \cong \mathbf{D}(s, s').$$

- (c) Conclude that for each  $m, n \geq 0$ , there are natural isomorphisms

$$\mathbf{sSet}(\Delta[m], \Delta[n]) \cong \Delta([m], [n]).$$

In other words, giving a map  $\xi_* : \Delta[m] \rightarrow \Delta[n]$  of simplicial sets is the same as giving a map  $\xi : [m] \rightarrow [n]$  in  $\Delta$ .

Recall from lecture the following definitions.

**Definition 2.** Let  $X$  be a simplicial set and  $Y$  a topological space.

- For each  $n \geq 0$ , the *standard topological  $n$ -simplex*  $\Delta^n$  is the subspace defined by

$$\Delta^n := \left\{ (t_0, \dots, t_n) \mid 0 \leq t_i \leq 1, \sum_{i=0}^n t_i = 1 \right\} \subseteq \mathbb{R}^{n+1}.$$

Define  $d^i : \Delta^{n-1} \rightarrow \Delta^n$  and  $s^j : \Delta^{n+1} \rightarrow \Delta^n$  for  $0 \leq i, j \leq n$  by

$$d^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}),$$

$$s^j(t_0, \dots, t_{n+1}) = (t_0, \dots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \dots, t_{n+1}).$$

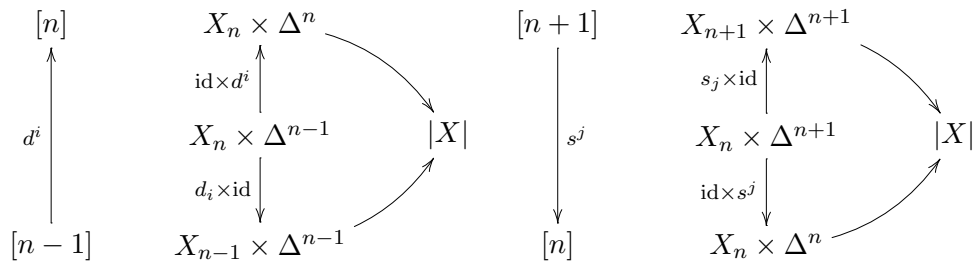
- For each  $n \geq 0$ , the *singular complex* of  $Y$ , denoted by  $S_\bullet(Y)$ , is the simplicial set with  $n$ -simplices defined by

$$S_n(Y) := \mathbf{Top}(\Delta^n, Y).$$

- The *geometric realization* of  $X$ , denoted by  $|X|$ , is the topological space defined by

$$|X| = \left( \prod_{n \geq 0} X_n \times \Delta^n \right) / \sim$$

modulo  $(d_i x, t) \sim (x, d^i t)$  and  $(s_j x, t) \sim (x, s^j t)$ , which may be pictured by the diagrams



It is easy to check that the cosimplicial identities (“dual” to the simplicial identities) are satisfied by the  $d^i$  and  $s^j$  maps which we picture here by the diagram

$$\begin{array}{ccccccc} & & \xleftarrow{s^0} & \xleftarrow{\quad} & \xleftarrow{\quad} & \xleftarrow{\quad} & \\ & & \xleftarrow{\quad} & \xleftarrow{\quad} & \xleftarrow{\quad} & \xleftarrow{\quad} & \\ \Delta^0 & \xrightarrow{d^0} & \Delta^1 & \xrightarrow{\quad} & \Delta^2 & \xrightarrow{\quad} & \Delta^3 \dots \\ & \xrightarrow{d^1} & & & & & \end{array}$$

in  $\mathbf{Top}$ . Hence, the topological  $n$ -simplices  $\Delta^n$  ( $n \geq 0$ ) determine a functor

$$\Delta^\bullet : \Delta \longrightarrow \mathbf{Top}, \quad [n] \longmapsto \Delta^n.$$

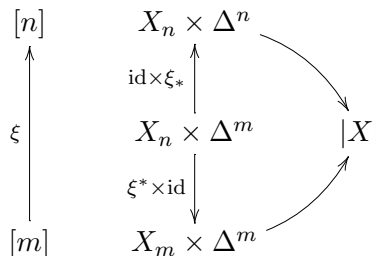
This will be useful for the following exercise.

**Exercise 4.** Let  $X$  be a simplicial set.

- (a) Prove that geometric realization  $|X|$  satisfies

$$|X| = \left( \prod_{n \geq 0} X_n \times \Delta^n \right) / \sim$$

modulo  $(\xi^* x, t) \sim (x, \xi_* t)$ , which may be pictured by the diagrams



- (b) Conclude that  $|X|$  satisfies the universal property: given any collection of maps  $\{f_n : X_n \times \Delta^n \rightarrow Z\}_{n \geq 0}$  such that the outer diagram in (1) commutes, then there exists a unique continuous map  $\bar{f}$  such that the diagram

$$(1) \quad \begin{array}{ccccc} & & X_n \times \Delta^n & & \\ & & \uparrow \text{id} \times \xi_* & \searrow f_n & \\ & [n] & X_n \times \Delta^m & \xrightarrow{\quad \bar{f} \quad} & Z \\ & \uparrow \xi & \downarrow \xi^* \times \text{id} & \xrightarrow{\quad \exists! \quad} & \\ & [m] & X_m \times \Delta^m & \xrightarrow{f_m} & \end{array}$$

in **Top** commutes for all maps  $\xi$  in  $\Delta$ .

- (c) Prove that  $|\Delta[k]|\cong \Delta^k$ .

For part (c), try looking for naturally occurring maps  $\{\Delta[k]_n \times \Delta^n \rightarrow \Delta^k\}_{n \geq 0}$  and verify the universal property of geometric realization.

*Remark 3.* Recall from lecture that geometric realization

$$|-| : \mathbf{sSet} \rightarrow \mathbf{Top}, \quad X \mapsto |X|$$

gives a well-defined functor. This can also be deduced directly from the universal property.

**Proposition 4.** *Let  $X$  be a simplicial set and  $Y$  a topological space. There are isomorphisms*

$$\mathbf{Top}(|X|, Y) \cong \mathbf{sSet}(X, S_\bullet(Y))$$

*natural in  $X, Y$ . In other words, there is an adjunction*

$$\mathbf{sSet} \begin{array}{c} \xrightarrow{|-|} \\ \xleftarrow{S_\bullet(-)} \end{array} \mathbf{Top}$$

*with left adjoint the geometric realization functor  $|-|$  and right adjoint the singular complex functor  $S_\bullet(-)$ .*

**Exercise 5.** Prove Proposition 4.

To get started on Exercise 5, consider any map  $f : X \rightarrow S_\bullet(Y)$  of simplicial sets, then this is the same as giving a collection of maps  $f_n : X_n \rightarrow S_n(Y)$  ( $n \geq 0$ ) in **Set** such that certain diagrams commute, which is the same as giving a corresponding collection of maps  $f_n : X_n \times \Delta^n \rightarrow Y$  ( $n \geq 0$ ) in **Top** such that certain diagrams commute.

In algebraic topology and homotopy theory, the existence of mapping spaces between any pair  $Y, Z$  of spaces is frequently useful; for example, loop spaces, path spaces, and homotopy limits are often defined or constructed using a topological space of maps. Similarly, when working with simplicial sets, it

is frequently useful to have available a simplicial set of maps between any pair  $Y, Z$  of simplicial sets.

**Definition 5.** Let  $Y, Z$  be simplicial sets. The *simplicial function complex*, denoted by  $\text{Map}(Y, Z)$ , is the simplicial set with  $n$ -simplices defined by

$$\text{Map}(Y, Z)_n := \mathbf{sSet}(Y \times \Delta[n], Z).$$

The following proposition is Exercise 6(b).

**Proposition 6.** Let  $X, Y, Z$  be simplicial sets. There are isomorphisms

$$\mathbf{sSet}(X \times Y, Z) \cong \mathbf{sSet}(X, \text{Map}(Y, Z))$$

natural in  $X, Y, Z$ . In particular, there is an adjunction

$$\mathbf{sSet} \begin{array}{c} \xleftarrow{-\times Y} \\ \xrightarrow{\text{Map}(Y, -)} \end{array} \mathbf{sSet}$$

with left adjoint the functor  $-\times Y$  and right adjoint the functor  $\text{Map}(Y, -)$ .

**Exercise 6.** Let  $X, Y, Z$  be simplicial sets. Prove the following.

- (a) Use Exercise 3(c) to define naturally occurring face and degeneracy maps for  $\text{Map}(X, Y)$  and verify these satisfy the simplicial identities; it is equivalent to use Exercise 3(c) to verify that  $\text{Map}(X, Y)$  is a well-defined functor  $\Delta^{\text{op}} \rightarrow \mathbf{Set}$ .
- (b) There are isomorphisms

$$\mathbf{sSet}(X \times Y, Z) \cong \mathbf{sSet}(X, \text{Map}(Y, Z))$$

natural in  $X, Y, Z$ .

- (c) There are isomorphisms

$$\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))$$

natural in  $X, Y, Z$ .

- (d)  $X \cong \text{Map}(*, X)$ ; here,  $* = \Delta[0]$ .

Parts (c) and (d) follow easily from (b) and the Yoneda Lemma (see, Exercise 3). For example, for (c) it suffices to verify that

$$\mathbf{sSet}(-, \text{Map}(X \times Y, Z)) \cong \mathbf{sSet}(-, \text{Map}(X, \text{Map}(Y, Z))).$$

Hence, it is enough to verify that there are natural isomorphisms

$$\mathbf{sSet}(A, \text{Map}(X \times Y, Z)) \cong \mathbf{sSet}(A, \text{Map}(X, \text{Map}(Y, Z))).$$

Here are some references for this material: [1, Section 3], [2, Chapter III], [3, Chapter 16].

## REFERENCES

- [1] W. G. Dwyer and H. Henn. *Homotopy theoretic methods in group cohomology*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2001.
- [2] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [3] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. Available at: <http://www.math.uchicago.edu/~may/> .