

Series 7

The purpose of the first part of this series is to develop some of the basic properties of the geometric realization  $|-|$  and singular complex  $S_\bullet$  functors, including the following proposition which is proved in Exercise 5.

**Proposition 1.** *Let  $X, Y$  be simplicial sets and  $n, m \geq 0$ . Assume that  $|Y|$  is locally compact Hausdorff.*

- (a) *The natural map  $|\Delta[n] \times \Delta[m]| \xrightarrow{\cong} |\Delta[n]| \times |\Delta[m]|$  is a homeomorphism.*
- (b) *The natural map  $|X \times \Delta[m]| \xrightarrow{\cong} |X| \times |\Delta[m]|$  is a homeomorphism.*
- (c) *The natural map  $|X \times Y| \xrightarrow{\cong} |X| \times |Y|$  is a homeomorphism.*

Let  $X$  be a simplicial set and consider the functor  $\Delta[-]$  defined objectwise by

$$\Delta \longrightarrow \mathbf{sSet}, \quad [n] \longmapsto \Delta[n].$$

The *category of simplices* of  $X$ , denoted  $\Delta[-] \downarrow X$ , is the category with objects all pairs  $([n], x)$  where  $[n] \in \Delta$  and  $x : \Delta[n] \rightarrow X$  is a map of simplicial sets, and with morphisms  $\xi : ([n], x) \rightarrow ([n'], x')$  those morphisms  $\xi : [n] \rightarrow [n']$  of  $\Delta$  such that  $x'\xi_* = x$ ; i.e., such that the diagram

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{\xi_*} & \Delta[n'] \\ x \downarrow & & \downarrow x' \\ X & \xlongequal{\quad} & X \end{array}$$

in  $\mathbf{sSet}$  commutes. The *diagram of simplices* of  $X$  is the functor defined objectwise by

$$\Delta[-] \downarrow X \longrightarrow \mathbf{sSet}, \quad (\Delta[n] \xrightarrow{x} X) \longmapsto \Delta[n].$$

The following proposition, proved in Exercise 2, shows that every simplicial set  $X$  can be constructed by gluing together (Exercise 1) standard  $n$ -simplices  $\Delta[n]$  according to the diagram of simplices of  $X$ .

**Proposition 2.** *Let  $X$  be a simplicial set. Then the natural map*

$$\operatorname{colim}_{\Delta[n] \rightarrow X} \Delta[n] \xrightarrow{\cong} X$$

*in  $\mathbf{sSet}$  is an isomorphism.*

Before proving this proposition, it will be useful to recall the universal property of colimits.

Let  $\mathbf{D}$  be a small category. Suppose  $X : \mathbf{D} \rightarrow \mathbf{sSet}$  is  $\mathbf{D}$ -shaped diagram in simplicial sets. The *colimit* of the diagram  $X$ , written  $\operatorname{colim} X$  (or

$\text{colim}_d X(d)$ , or  $\text{colim}_{\mathbf{D}} X$ , or  $\varinjlim X$ , is a simplicial set, together with a collection of maps  $\{i_d : X(d) \rightarrow \text{colim } X\}_{d \in \mathbf{D}}$  such that the diagrams

$$(1) \quad (\xi \text{ in } \mathbf{D}) \quad \begin{array}{ccc} d & X(d) & \\ \downarrow \xi & \downarrow X(\xi) & \searrow i_d \\ d' & X(d') & \text{colim } X \\ & \uparrow i_{d'} & \end{array}$$

in  $\mathbf{sSet}$  commute, which satisfies the universal property: given a simplicial set  $Y$  together with a collection of maps  $\{g_d : X(d) \rightarrow Y\}_{d \in \mathbf{D}}$  such that the outer diagrams in (2) commute, then there exists a unique map  $\bar{g}$  of simplicial sets such that the diagrams

$$(2) \quad (\xi \text{ in } \mathbf{D}) \quad \begin{array}{ccccc} d & X(d) & & & \\ \downarrow \xi & \downarrow X(\xi) & \xrightarrow{g_d} & & \\ d' & X(d') & \xrightarrow{g_{d'}} & & \\ & & & \text{colim } X & \xrightarrow{\bar{g}} Y \\ & & & \exists! & \end{array}$$

in  $\mathbf{sSet}$  commute.

The following exercise shows that the colimit of a  $\mathbf{D}$ -shaped diagram can be calculated using coproducts and quotients. The construction is suggested by the diagrammatic formulation in (2) of the universal property of colimits.

**Exercise 1.** Let  $\mathbf{D}$  be a small category and consider any  $\mathbf{D}$ -shaped diagram  $X : \mathbf{D} \rightarrow \mathbf{sSet}$ . Prove the following.

$$\text{colim } X \cong \left( \coprod_{d \in \mathbf{D}} X(d) \right) / \sim$$

Here,  $\sim$  denotes the equivalence relation generated by  $x \sim X(\xi)(x)$ ,  $x \in X(d)$ , for every map  $\xi : d \rightarrow d'$  in  $\mathbf{D}$ ; see diagram (2). More informally,  $\text{colim } X$  can be built by gluing together the objects  $X(d)$  ( $d \in \mathbf{D}$ ); the gluing data is encoded by the shape of  $\mathbf{D}$ .

**Exercise 2.** Prove Proposition 2.

To get started on Exercise 2, note that it is enough to verify the universal property of colimits: given a simplicial set  $Y$  together with a collection of maps  $\{g_{([n],x)} : \Delta[n] \rightarrow Y\}_{([n],x) \in \Delta[-] \downarrow X}$  such that the right-hand outer diagrams in (3) commute, then there exists a unique map  $\bar{g}$  of simplicial

sets such that the right-hand diagrams

$$(3) \quad \begin{array}{ccc} [n] & (\Delta[n] \xrightarrow{x} X) & \Delta[n] \\ \downarrow \xi & \downarrow \xi_* & \downarrow \xi_* \\ [n'] & (\Delta[n'] \xrightarrow{x'} X) & \Delta[n'] \end{array} \quad \begin{array}{ccc} & & \Delta[n] \\ & & \downarrow \xi_* \\ & & X \\ & & \downarrow \exists! \\ & & Y \end{array} \quad \begin{array}{ccc} & & \Delta[n] \\ & & \downarrow \xi_* \\ & & X \\ & & \downarrow \exists! \\ & & Y \end{array}$$

(ξ in Δ)

in  $\mathbf{sSet}$  commute. The natural isomorphisms  $\mathbf{sSet}(\Delta[n], Y) \cong Y_n$  from [Series 6, Exercise 2] will be useful.

Recall the following.

**Proposition 3.** *Let  $X$  be a simplicial set and  $Y$  a topological space. There are isomorphisms*

$$(4) \quad \mathbf{Top}(|X|, Y) \cong \mathbf{sSet}(X, S_\bullet(Y))$$

natural in  $X, Y$ . In particular, the left-hand diagram in (5) commutes

$$(5) \quad \begin{array}{ccc} |X| & \xrightarrow{g} & Y \\ |p| \downarrow & & \downarrow q \\ |X'| & \xrightarrow{h} & Y' \end{array} \quad \begin{array}{ccc} X & \xrightarrow{g^\#} & S_\bullet(Y) \\ p \downarrow & & \downarrow S_\bullet(q) \\ X' & \xrightarrow{h^\#} & S_\bullet(Y') \end{array}$$

if and only if the corresponding right-hand diagram in (5) commutes. Here, the maps  $g : |X| \rightarrow Y$  and  $g^\# : X \rightarrow S_\bullet(Y)$  correspond under the natural isomorphisms (4), and similarly for the maps  $h$  and  $h^\#$ .

The following proposition, which is proved in Exercise 3, is an important property of the geometric realization functor.

**Proposition 4.** *Let  $\mathbf{D}$  be a small category and consider any  $\mathbf{D}$ -shaped diagram  $X : \mathbf{D} \rightarrow \mathbf{sSet}$ . The geometric realization functor  $|-| : \mathbf{sSet} \rightarrow \mathbf{Top}$  preserves colimit diagrams. In particular, there are isomorphisms*

$$|\operatorname{colim} X| \cong \operatorname{colim} |X|$$

natural in  $X$ . In other words, the geometric realization functor  $|-|$  commutes with colimits.

**Exercise 3.** Use Proposition 3 to prove Proposition 4.

Recall the following; see also Definition 7 below.

**Proposition 5.** *Let  $n \geq 0$ . There are homeomorphisms*

$$|\Delta[n]| \cong \Delta^n, \quad |\partial\Delta[n]| \cong \partial\Delta^n.$$

**Exercise 4.** Let  $n \geq 1$ . Use Propositions 4 and 5 to prove that

$$|\Delta[n]/\partial\Delta[n]| \cong \Delta^n/\partial\Delta^n \cong S^n.$$

**Proposition 6.** *Let  $A, B, C$  be simplicial sets (resp. topological spaces such that  $B$  is locally compact Hausdorff). Let  $\mathbf{D}$  be a small category and consider any  $\mathbf{D}$ -shaped diagram  $X : \mathbf{D} \rightarrow \mathbf{sSet}$  (resp.  $X : \mathbf{D} \rightarrow \mathbf{Top}$ ). There are isomorphisms*

$$\begin{aligned} \mathbf{sSet}(A \times B, C) &\cong \mathbf{sSet}(A, \text{Map}(B, C)) \\ \mathbf{Top}(A \times B, C) &\cong \mathbf{Top}(A, \text{Map}(B, C)) \end{aligned}$$

natural in such  $A, B, C$ . In particular, there are isomorphisms

$$(\text{colim } X) \times B \cong \text{colim}(X \times B)$$

natural in  $X$ ; in other words, the functor  $- \times B$  commutes with colimits.

**Exercise 5.** Assume Proposition 1(a); a proof can be found in [1, Section 3] and [4, Chapter 2].

- (a) Verify the details in the proof of Proposition 1(b) below.
- (b) Use a similar argument to prove Proposition 1(c).

*Proof of Proposition 1(b).* There are natural isomorphisms

$$\begin{aligned} |X \times \Delta[m]| &\cong \left| \left( \text{colim}_{\Delta[n] \rightarrow X} \Delta[n] \right) \times \Delta[m] \right| \cong \left| \text{colim}_{\Delta[n] \rightarrow X} \left( \Delta[n] \times \Delta[m] \right) \right| \\ &\cong \text{colim}_{\Delta[n] \rightarrow X} \left| \left( \Delta[n] \times \Delta[m] \right) \right| \cong \text{colim}_{\Delta[n] \rightarrow X} \left( |\Delta[n]| \times |\Delta[m]| \right) \\ &\cong \left( \text{colim}_{\Delta[n] \rightarrow X} |\Delta[n]| \right) \times |\Delta[m]| \cong \left| \text{colim}_{\Delta[n] \rightarrow X} \Delta[n] \right| \times |\Delta[m]| \\ &\cong |X| \times |\Delta[m]|. \end{aligned}$$

□

**Exercise 6.** Let  $X$  and  $Y$  be simplicial sets. Prove the following.

- (a)  $S_\bullet(X) \times S_\bullet(Y) \cong S_\bullet(X \times Y)$ .
- (b)  $S_\bullet$  commutes with limits.

To get started on (b), dualize the argument in Exercise 3.

Recall from lecture the following subcomplexes of  $\Delta[n]$ .

**Definition 7.** Denote by  $\text{id}_{[n]}$  the non-degenerate  $n$ -simplex of  $\Delta[n]$  ( $n \geq 0$ ).

- For  $n \geq 0$ , denote by  $\partial\Delta[n] \subset \Delta[n]$  the largest subcomplex not containing the  $n$ -simplex  $\text{id}_{[n]}$ ; i.e.,

$$(\partial\Delta[n])_m = \{[m] \xrightarrow{\xi} [n] \in \Delta \mid \text{image}(\xi) \not\supseteq [n]\}.$$

- For ( $n \geq 1, 0 \leq k \leq n$ ), denote by  $\Lambda^k[n] \subset \Delta[n]$  the largest subcomplex not containing the  $(n-1)$ -simplex  $d_k \text{id}_{[n]}$ ; i.e.,

$$(\Lambda^k[n])_m = \{[m] \xrightarrow{\xi} [n] \in \Delta \mid \text{image}(\xi) \not\supseteq [n] - \{k\}\}.$$

A map  $p : X \rightarrow Y$  of simplicial sets is a *Kan fibration* (or fibration) if it has the right lifting property with respect to the inclusions  $\Lambda^k[n] \rightarrow \Delta[n]$  ( $n \geq 1, 0 \leq k \leq n$ ); i.e., if given a solid commutative diagram of the form

$$(6) \quad \begin{array}{ccc} \Lambda^k[n] & \longrightarrow & X \\ \downarrow & \nearrow x & \downarrow p \\ \Delta[n] & \longrightarrow & Y \end{array} \quad \exists$$

in  $\mathbf{sSet}$ , then there exists a map  $x$  of simplicial sets such that the diagram (6) commutes.

**Exercise 7.** Prove the following:

- Every isomorphism is a Kan fibration.
- The composition of two Kan fibrations is a Kan fibration.
- The product of two Kan fibrations is a Kan fibration.

A simplicial set  $X$  is a *Kan complex* (or is fibrant) if the map  $X \rightarrow *$  is a Kan fibration; i.e., if given a solid commutative diagram of the form

$$(7) \quad \begin{array}{ccc} \Lambda^k[n] & \longrightarrow & X \\ \downarrow & \nearrow x & \downarrow \\ \Delta[n] & \longrightarrow & * \end{array} \quad \exists$$

in  $\mathbf{sSet}$  with ( $n \geq 1, 0 \leq k \leq n$ ), then there exists a map  $x$  of simplicial sets such that the diagram (7) commutes. Here,  $*$  denotes the simplicial set  $\Delta[0]$ .

**Proposition 8.** Let  $n \geq 1$  and  $0 \leq k \leq n$ . Then  $|\Lambda^k[n]| \subset |\Delta[n]|$  is a strong deformation retract.

**Exercise 8.** Let  $Z$  be a topological space. Use Proposition 8 to prove that the singular complex  $S_\bullet(Z)$  is a Kan complex.

Denote by  $\mathbf{Grp}$  the category of groups and their homomorphisms. A *simplicial group* is a  $\Delta^{\text{op}}$ -shaped diagram  $X : \Delta^{\text{op}} \rightarrow \mathbf{Grp}$  and a *map of simplicial groups* is a natural transformation  $f : X \rightarrow X'$ . In other words, a simplicial group  $X$  is a diagram of the form

$$\begin{array}{ccccccc} & & & & \rightrightarrows & & \\ & & & & \rightrightarrows & & \\ & & & & \rightrightarrows & & \\ \xrightarrow{s_0} & & \rightrightarrows & & \rightrightarrows & & \\ X_0 & \xleftarrow{d_0} & X_1 & \xleftarrow{d_1} & X_2 & \xleftarrow{d_2} & X_3 \cdots \end{array}$$

in **Grp** such that the simplicial identities

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, & i < j, \\ s_i s_j &= s_{j+1} s_i, & i \leq j, \\ d_i s_j &= \begin{cases} s_{j-1} d_i, & i < j, \\ \text{id}, & i = j, \\ s_j d_{i-1}, & i > j + 1 \end{cases} & i = j + 1 \end{aligned}$$

are satisfied.

**Theorem 9.** *If  $X$  is a simplicial group, then the underlying simplicial set of  $X$  is fibrant.*

**Exercise 9.** Let  $X$  be a simplicial group. The purpose of this exercise is to prove Theorem 9 by verifying the extension conditions in (7). It is useful to first consider the low dimensional cases.

- (a) Let  $n = 2$  and  $k = 2$ . Consider a 2-tuple  $(x_0, x_1)$  of 1-simplices of  $X$  which satisfies the compatibility condition that  $d_0 x_1 = d_0 x_0$  as indicated in the left-hand picture

(8)

- Prove the following: there exists a 2-simplex  $x$  such that  $d_0 x = x_0$  and  $d_1 x = x_1$  as indicated in the right-hand picture in (8).

To get started, define

$$u^0 := s_0 x_0, \quad y^0 := s_1((d_1 u^0)^{-1} x_1), \quad x := u^1 := u^0 y^0$$

and use the simplicial identities together with the compatibility condition to verify that  $d_0 x = x_0$  and  $d_1 x = x_1$ .

- (b) Let  $n = 2$  and  $k = 1$ . Consider a 2-tuple  $(x_0, x_2)$  of 1-simplices of  $X$  which satisfies the compatibility condition that  $d_1 x_0 = d_0 x_2$  as indicated in the left-hand picture

(9)

- Prove the following: there exists a 2-simplex  $x$  such that  $d_0 x = x_0$  and  $d_2 x = x_2$  as indicated in the right-hand picture in (9).

To get started, define

$$\begin{aligned} u^0 &:= s_0 x_0, & v^0 &:= u^0 = s_0 x_0, & z^0 &:= s_1((d_2 v^0)^{-1} x_2), \\ x &:= v^1 := v^0 z^0 = s_0(x_0) s_1((d_2 s_0 x_0)^{-1} x_2) \end{aligned}$$

and use the simplicial identities together with the compatibility condition to verify that  $d_0x = x_0$  and  $d_2x = x_2$ .

- (c) Let  $n = 2$  and  $k = 0$ . Consider a 2-tuple  $(x_1, x_2)$  of 1-simplices of  $X$  which satisfies the compatibility condition that  $d_1x_1 = d_1x_2$  as indicated in the left-hand picture

$$(10) \quad \begin{array}{ccc} & 2 & \\ x_1 \nearrow & & \\ 0 \xrightarrow{x_2} & & 1 \end{array} \qquad \begin{array}{ccc} & 2 & \\ x_1 \nearrow & & d_0x \nwarrow \\ 0 \xrightarrow{x_2} & & 1, \end{array}$$

- Prove the following: there exists a 2-simplex  $x$  such that  $d_1x = x_1$  and  $d_2x = x_2$  as indicated in the right-hand picture in (10). To get started, define

$$\begin{aligned} v^0 &:= e, & z^0 &:= s_1((d_2v^0)^{-1}x_2) = s_1x_2, \\ v^1 &:= v^0z^0 = s_1x_2, & z^1 &:= s_0((d_1v^1)^{-1}x_1) = s_0(x_2^{-1}x_1), \\ x &:= v^2 := v^1z^1 = s_1(x_2)s_0(x_2^{-1}x_1) \end{aligned}$$

and use the simplicial identities together with the compatibility condition to verify that  $d_1x = x_1$  and  $d_2x = x_2$ . Here,  $e$  denotes the unit element of a group.

- (d) To complete the proof for the remaining  $n$  and  $k$ , see [6, Theorem 17.1] which uses the same notation as above. See also [3, Chapter I].

The extension condition (7) has the following combinatorial reformulation.

**Proposition 10.** *Let  $X$  be a simplicial set. The following are equivalent:*

- $X$  is a Kan complex.
- $X$  satisfies the following extension condition: if given an  $n$ -tuple  $(x_0, \dots, \hat{x}_k, \dots, x_n)$  of  $(n-1)$ -simplices of  $X$  with  $(n \geq 1, 0 \leq k \leq n)$  and which satisfies the compatibility condition that  $d_ix_j = d_{j-1}x_i$  for  $i < j$  and  $i, j \neq k$ , then there exists an  $n$ -simplex  $x$  such that  $d_ix = x_i$  for  $i \neq k$ .

Here, the notation  $(x_0, \dots, \hat{x}_k, \dots, x_n)$  means that  $x_k$  is not there.

Here are some references for this material: [1, Section 3], [2, Chapters II and III], [3, Chapter I], [4, Chapter 2], [5, Chapter III.3 and III.4], [6, Section 17], [7, Chapter 2.6].

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