Homology and cohomology **Spring 2009**
Prof. Kathryn Hess

Series 7

The purpose of the first part of this series is to develop some of the basic properties of the geometric realization $|-|$ and singular complex $S_\bullet$ functors, including the following proposition which is proved in Exercise 5.

Proposition 1. *Let X, Y be simplicial sets and $n, m \geq 0$. Assume that $|Y|$ is locally compact Hausdorff.*

(a) *The natural map $|\Delta[n] \times \Delta[m]| \xrightarrow{\cong} |\Delta[n]| \times |\Delta[m]|$ is a homeomorphism.*
(b) *The natural map $|X \times \Delta[m]| \xrightarrow{\cong} |X| \times |\Delta[m]|$ is a homeomorphism.*
(c) *The natural map $|X \times Y| \xrightarrow{\cong} |X| \times |Y|$ is a homeomorphism.*

Let X be a simplicial set and consider the functor $\Delta[-]$ defined objectwise by

$$\Delta \longrightarrow \mathbf{sSet}, \qquad [n] \longmapsto \Delta[n].$$

The *category of simplices* of X, denoted $\Delta[-] \downarrow X$, is the category with objects all pairs $([n], x)$ where $[n] \in \Delta$ and $x : \Delta[n] \longrightarrow X$ is a map of simplicial sets, and with morphisms $\xi : ([n], x) \longrightarrow ([n'], x')$ those morphisms $\xi : [n] \longrightarrow [n']$ of Δ such that $x'\xi_* = x$; i.e., such that the diagram

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{\xi_*} & \Delta[n'] \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x'} \\ X & = & X \end{array}$$

in **sSet** commutes. The *diagram of simplices* of X is the functor defined objectwise by

$$\Delta[-] \downarrow X \longrightarrow \mathbf{sSet}, \qquad (\Delta[n] \xrightarrow{x} X) \longmapsto \Delta[n].$$

The following proposition, proved in Exercise 2, shows that every simplicial set X can be constructed by gluing together (Exercise 1) standard n-simplices $\Delta[n]$ according to the diagram of simplices of X.

Proposition 2. *Let X be a simplicial set. Then the natural map*

$$\operatorname*{colim}_{\Delta[n] \to X} \Delta[n] \xrightarrow{\cong} X$$

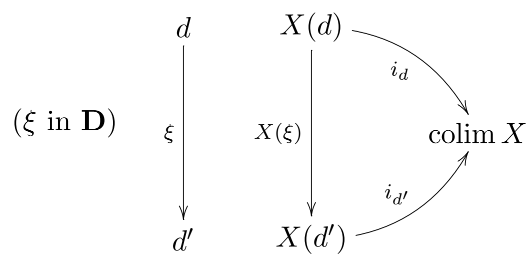
in **sSet** *is an isomorphism.*

Before proving this proposition, it will be useful to recall the universal property of colimits.

Let $\mathbf{D}$ be a small category. Suppose $X : \mathbf{D} \longrightarrow \mathbf{sSet}$ is $\mathbf{D}$-shaped diagram in simplicial sets. The *colimit* of the diagram X, written $\operatorname{colim} X$ (or

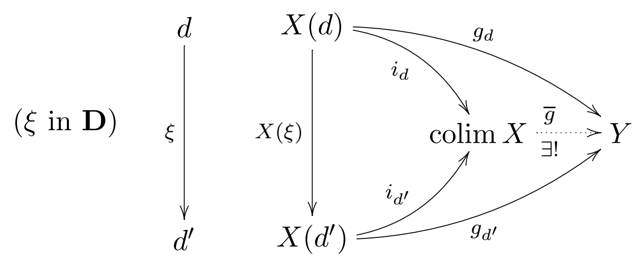
$\mathrm{colim}_d\, X(d)$, or $\mathrm{colim}_{\mathbf{D}}\, X$, or $\varinjlim X$), is a simplicial set, together with a collection of maps $\{i_d : X(d) \longrightarrow \mathrm{colim}\, X\}_{d \in \mathbf{D}}$ such that the diagrams

(1)



in **sSet** commute, which satisfies the universal property: given a simplicial set Y together with a collection of maps $\{g_d : X(d) \longrightarrow Y\}_{d \in \mathbf{D}}$ such that the outer diagrams in (2) commute, then there exists a unique map $\overline{g}$ of simplicial sets such that the diagrams

(2)



in **sSet** commute.

The following exercise shows that the colimit of a **D**-shaped diagram can be calculated using coproducts and quotients. The construction is suggested by the diagrammatic formulation in (2) of the universal property of colimits.

Exercise 1. Let **D** be a small category and consider any **D**-shaped diagram $X : \mathbf{D} \longrightarrow \mathbf{sSet}$. Prove the following.

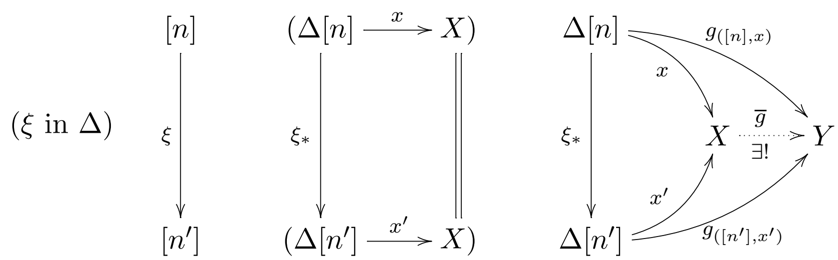
$$\mathrm{colim}\, X \cong \Big(\coprod_{d \in \mathbf{D}} X(d) \Big) / \sim$$

Here, $\sim$ denotes the equivalence relation generated by $x \sim X(\xi)(x)$, $x \in X(d)$, for every map $\xi : d \longrightarrow d'$ in **D**; see diagram (2). More informally, $\mathrm{colim}\, X$ can be built by gluing together the objects $X(d)$ ($d \in \mathbf{D}$); the gluing data is encoded by the shape of **D**.

Exercise 2. Prove Proposition 2.

To get started on Exercise 2, note that it is enough to verify the universal property of colimits: given a simplicial set Y together with a collection of maps $\{g_{([n],x)} : \Delta[n] \longrightarrow Y\}_{([n],x) \in \Delta[-] \downarrow X}$ such that the right-hand outer diagrams in (3) commute, then there exists a unique map $\overline{g}$ of simplicial sets such that the right-hand diagrams

(3)



in **sSet** commute. The natural isomorphisms $\mathbf{sSet}(\Delta[n], Y) \cong Y_n$ from [Series 6, Exercise 2] will be useful.

Recall the following.

Proposition 3. *Let X be a simplicial set and Y a topological space. There are isomorphisms*

$$\mathbf{Top}(|X|, Y) \cong \mathbf{sSet}(X, S_\bullet(Y)) \tag{4}$$

natural in X, Y. In particular, the left-hand diagram in (5) commutes

$$\begin{array}{ccc} |X| & \xrightarrow{g} & Y \\ {\scriptstyle |p|}\downarrow & & \downarrow{\scriptstyle q} \\ |X'| & \xrightarrow{h} & Y' \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{g^\#} & S_\bullet(Y) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle S_\bullet(q)} \\ X' & \xrightarrow{h^\#} & S_\bullet(Y') \end{array} \tag{5}$$

if and only if the corresponding right-hand diagram in (5) commutes. Here, the maps $g : |X| \longrightarrow Y$ and $g^\# : X \longrightarrow S_\bullet(Y)$ correspond under the natural isomorphisms (4), and similarly for the maps h and $h^\#$.

The following proposition, which is proved in Exercise 3, is an important property of the geometric realization functor.

Proposition 4. *Let $\mathbf{D}$ be a small category and consider any $\mathbf{D}$-shaped diagram $X : \mathbf{D} \longrightarrow \mathbf{sSet}$. The geometric realization functor $|-| : \mathbf{sSet} \longrightarrow \mathbf{Top}$ preserves colimit diagrams. In particular, there are isomorphisms*

$$|\operatorname{colim} X| \cong \operatorname{colim} |X|$$

natural in X. In other words, the geometric realization functor $|-|$ commutes with colimits.

Exercise 3. Use Proposition 3 to prove Proposition 4.

Recall the following; see also Definition 7 below.

Proposition 5. *Let $n \geq 0$. There are homeomorphisms*

$$|\Delta[n]| \cong \Delta^n, \qquad |\partial\Delta[n]| \cong \partial\Delta^n.$$

Exercise 4. Let $n \geq 1$. Use Propositions 4 and 5 to prove that

$$\big|\Delta[n]/\partial\Delta[n]\big| \cong \Delta^n/\partial\Delta^n \cong S^n.$$

Proposition 6. *Let A, B, C be simplicial sets (resp. topological spaces such that B is locally compact Hausdorff). Let* $\mathbf{D}$ *be a small category and consider any* $\mathbf{D}$*-shaped diagram* $X : \mathbf{D} \longrightarrow \mathbf{sSet}$ *(resp.* $X : \mathbf{D} \longrightarrow \mathbf{Top}$*). There are isomorphisms*

$$\mathbf{sSet}(A \times B, C) \cong \mathbf{sSet}(A, \mathrm{Map}(B, C))$$
$$\mathbf{Top}(A \times B, C) \cong \mathbf{Top}(A, \mathrm{Map}(B, C))$$

natural in such A, B, C. In particular, there are isomorphisms

$$\big(\mathrm{colim}\, X\big) \times B \cong \mathrm{colim}(X \times B)$$

natural in X; in other words, the functor $- \times B$ commutes with colimits.

Exercise 5. Assume Proposition 1(a); a proof can be found in [1, Section 3] and [4, Chapter 2].

(a) Verify the details in the proof of Proposition 1(b) below.
(b) Use a similar argument to prove Proposition 1(c).

Proof of Proposition 1(b). There are natural isomorphisms

$$\begin{aligned}
|X \times \Delta[m]| &\cong \Big|\Big(\operatorname*{colim}_{\Delta[n]\to X} \Delta[n]\Big) \times \Delta[m]\Big| \cong \Big|\operatorname*{colim}_{\Delta[n]\to X} \Big(\Delta[n] \times \Delta[m]\Big)\Big| \\
&\cong \operatorname*{colim}_{\Delta[n]\to X} \Big|\Big(\Delta[n] \times \Delta[m]\Big)\Big| \cong \operatorname*{colim}_{\Delta[n]\to X} \Big(|\Delta[n]| \times |\Delta[m]|\Big) \\
&\cong \Big(\operatorname*{colim}_{\Delta[n]\to X} |\Delta[n]|\Big) \times |\Delta[m]| \cong \Big|\operatorname*{colim}_{\Delta[n]\to X} \Delta[n]\Big| \times |\Delta[m]| \\
&\cong |X| \times |\Delta[m]|.
\end{aligned}$$

□

Exercise 6. Let X and Y be simplicial sets. Prove the following.

(a) $S_\bullet(X) \times S_\bullet(Y) \cong S_\bullet(X \times Y)$.
(b) $S_\bullet$ commutes with limits.

To get started on (b), dualize the argument in Exercise 3.

Recall from lecture the following subcomplexes of $\Delta[n]$.

Definition 7. Denote by $\mathrm{id}_{[n]}$ the non-degenerate n-simplex of $\Delta[n]$ ($n \geq 0$).

- For $n \geq 0$, denote by $\partial\Delta[n] \subset \Delta[n]$ the largest subcomplex not containing the n-simplex $\mathrm{id}_{[n]}$; i.e.,

$$(\partial\Delta[n])_m = \big\{[m] \xrightarrow{\xi} [n] \in \Delta \,|\, \text{ image}(\xi) \not\supset [n]\big\}.$$

- For $(n \geq 1,\, 0 \leq k \leq n)$, denote by $\Lambda^k[n] \subset \Delta[n]$ the largest subcomplex not containing the $(n-1)$-simplex $d_k\mathrm{id}_{[n]}$; i.e.,

$$(\Lambda^k[n])_m = \big\{[m] \xrightarrow{\xi} [n] \in \Delta \,|\, \text{ image}(\xi) \not\supset [n] - \{k\}\,\big\}.$$

A map $p : X \longrightarrow Y$ of simplicial sets is a *Kan fibration* (or fibration) if it has the right lifting property with respect to the inclusions $\Lambda^k[n] \longrightarrow \Delta[n]$ ($n \geq 1$, $0 \leq k \leq n$); i.e., if given a solid commutative diagram of the form

(6)
$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & X \\ \downarrow & \overset{x}{\nearrow}\ \exists & \downarrow p \\ \Delta[n] & \longrightarrow & Y \end{array}$$

in **sSet**, then there exists a map x of simplicial sets such that the diagram (6) commutes.

Exercise 7. Prove the following:

(a) Every isomorphism is a Kan fibration.
(b) The composition of two Kan fibrations is a Kan fibration.
(c) The product of two Kan fibrations is a Kan fibration.

A simplicial set X is a *Kan complex* (or is fibrant) if the map $X \longrightarrow *$ is a Kan fibration; i.e., if given a solid commutative diagram of the form

(7)
$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & X \\ \downarrow & \overset{x}{\nearrow}\ \exists & \downarrow \\ \Delta[n] & \longrightarrow & * \end{array}$$

in **sSet** with ($n \geq 1$, $0 \leq k \leq n$), then there exists a map x of simplicial sets such that the diagram (7) commutes. Here, $*$ denotes the simplicial set $\Delta[0]$.

Proposition 8. *Let $n \geq 1$ and $0 \leq k \leq n$. Then $|\Lambda^k[n]| \subset |\Delta[n]|$ is a strong deformation retract.*

Exercise 8. Let Z be a topological space. Use Proposition 8 to prove that the singular complex $S_\bullet(Z)$ is a Kan complex.

Denote by **Grp** the category of groups and their homomorphisms. A *simplicial group* is a Δ^{op}-shaped diagram $X : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Grp}$ and a *map of simplicial groups* is a natural transformation $f : X \longrightarrow X'$. In other words, a simplicial group X is a diagram of the form

$$X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1 \quad X_1 \leftleftarrows\!\!\leftarrow X_2 \quad X_2 \leftleftarrows\!\!\leftleftarrows X_3 \cdots \qquad (s_0 : X_0 \rightarrow X_1,\ X_1 \rightrightarrows X_2,\ X_2 \rightrightarrows\!\!\rightarrow X_3)$$

in **Grp** such that the simplicial identities

$$d_i d_j = d_{j-1} d_i, \qquad i < j,$$
$$s_i s_j = s_{j+1} s_i, \qquad i \leq j,$$
$$d_i s_j = \begin{cases} s_{j-1} d_i, & i < j, \\ \text{id}, & i = j, \quad i = j+1 \\ s_j d_{i-1}, & i > j+1 \end{cases}$$

are satisfied.

Theorem 9. *If X is a simplicial group, then the underlying simplicial set of X is fibrant.*

Exercise 9. Let X be a simplicial group. The purpose of this exercise is to prove Theorem 9 by verifying the extension conditions in (7). It is useful to first consider the low dimensional cases.

(a) Let $n = 2$ and $k = 2$. Consider a 2-tuple (x_0, x_1) of 1-simplices of X which satisfies the compatibility condition that $d_0 x_1 = d_0 x_0$ as indicated in the left-hand picture

(8)

- Prove the following: there exists a 2-simplex x such that $d_0 x = x_0$ and $d_1 x = x_1$ as indicated in the right-hand picture in (8).

To get started, define

$$u^0 := s_0 x_0, \quad y^0 := s_1\big((d_1 u^0)^{-1} x_1\big), \quad x := u^1 := u^0 y^0$$

and use the simplicial identities together with the compatibility condition to verify that $d_0 x = x_0$ and $d_1 x = x_1$.

(b) Let $n = 2$ and $k = 1$. Consider a 2-tuple (x_0, x_2) of 1-simplices of X which satisfies the compatibility condition that $d_1 x_0 = d_0 x_2$ as indicated in the left-hand picture

(9)

- Prove the following: there exists a 2-simplex x such that $d_0 x = x_0$ and $d_2 x = x_2$ as indicated in the right-hand picture in (9).

To get started, define

$$u^0 := s_0 x_0, \quad v^0 := u^0 = s_0 x_0, \quad z^0 := s_1\big((d_2 v^0)^{-1} x_2\big),$$
$$x := v^1 := v^0 z^0 = s_0(x_0) s_1\big((d_2 s_0 x_0)^{-1} x_2\big)$$

and use the simplicial identities together with the compatibility condition to verify that $d_0 x = x_0$ and $d_2 x = x_2$.

(c) Let $n = 2$ and $k = 0$. Consider a 2-tuple (x_1, x_2) of 1-simplices of X which satisfies the compatibility condition that $d_1 x_1 = d_1 x_2$ as indicated in the left-hand picture

(10)

- Prove the following: there exists a 2-simplex x such that $d_1 x = x_1$ and $d_2 x = x_2$ as indicated in the right-hand picture in (10).

To get started, define

$$v^0 := e, \quad z^0 := s_1\big((d_2 v^0)^{-1} x_2\big) = s_1 x_2,$$
$$v^1 := v^0 z^0 = s_1 x_2, \quad z^1 := s_0\big((d_1 v^1)^{-1} x_1\big) = s_0(x_2^{-1} x_1),$$
$$x := v^2 := v^1 z^1 = s_1(x_2) s_0(x_2^{-1} x_1)$$

and use the simplicial identities together with the compatibility condition to verify that $d_1 x = x_1$ and $d_2 x = x_2$. Here, e denotes the unit element of a group.

(d) To complete the proof for the remaining n and k, see [6, Theorem 17.1] which uses the same notation as above. See also [3, Chapter I].

The extension condition (7) has the following combinatorial reformulation.

Proposition 10. *Let X be a simplicial set. The following are equivalent:*

(a) *X is a Kan complex.*

(b) *X satisfies the following extension condition: if given an n-tuple $(x_0, \cdots, \hat{x}_k, \cdots, x_n)$ of $(n-1)$-simplices of X with ($n \geq 1$, $0 \leq k \leq n$) and which satisfies the compatibility condition that $d_i x_j = d_{j-1} x_i$ for $i < j$ and $i, j \neq k$, then there exists an n-simplex x such that $d_i x = x_i$ for $i \neq k$.*

Here, the notation $(x_0, \cdots, \hat{x}_k, \cdots, x_n)$ means that x_k is not there.

Here are some references for this material: [1, Section 3], [2, Chapters II and III], [3, Chapter I], [4, Chapter 2], [5, Chapter III.3 and III.4], [6, Section 17], [7, Chapter 2.6].

References

[1] W. G. Dwyer and H. Henn. *Homotopy theoretic methods in group cohomology.* Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2001. Available at (paper 96): `http://www.nd.edu/~wgd/` .

[2] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[3] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[4] D. R. Grayson. *Algebraic K-theory*. 2003. Available at: `http://www.math.uiuc.edu/~dan/Courses/2003/Spring/416/` .

[5] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[6] J. P. May. *Simplicial objects in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992. Reprint of the 1967 original.

[7] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. Available at: `http://www.math.uchicago.edu/~may/` .