Homology and cohomology **Spring 2009**
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Series 8

Let Z be a non-empty topological space, consider the natural map $p : Z \longrightarrow *$, and note that $C(*) \cong \mathbb{Z}$; here $\mathbb{Z}$ denotes the chain complex concentrated at 0 with value $\mathbb{Z}$. Hence $C(Z)$ is an augmented chain complex. The *reduced chain complex* $\tilde{C}(Z)$ of the augmented chain complex $C(Z)$ is defined by the short exact sequence

$$0 \to \tilde{C}(Z) \to C(Z) \xrightarrow{p_*} C(*) \to 0$$

of chain complexes. The *reduced homology* of Z, denoted $\tilde{H}(Z)$, is defined by $\tilde{H}_n(Z) := H_n(\tilde{C}(Z))$.

Exercise 1. Let Z be a non-empty topological space. Prove the following.

(a) $H_n(Z) \cong \tilde{H}_n(Z)$, for all $n \geq 1$.
(b) There is a short exact sequence

$$0 \to \tilde{H}_0(Z) \to H_0(Z) \to \mathbb{Z} \to 0$$

of abelian groups.
(c) $H_0(Z) \cong \tilde{H}_0(Z) \oplus \mathbb{Z}$.
(d) $\tilde{H}_n(S^0) = 0$, for all $n \geq 1$.
(e) $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$.
(f) $\tilde{H}_0(S^0) \cong \mathbb{Z}$.

The following is the reduced version of the Mayer-Vietoris sequence.

Theorem 1 (Reduced Mayer-Vietoris Theorem)**.** *Let Z be a topological space and suppose $X, Y \subset Z$ are subspaces such that $Z = \mathrm{int}(X) \cup \mathrm{int}(Y)$. Assume $X \cap Y \neq \emptyset$. Consider the commutative diagram of inclusions*

$$\begin{array}{ccc} X \cap Y & \xrightarrow{i^Y} & Y \\ {\scriptstyle i^X}\downarrow & & \downarrow{\scriptstyle j^Y} \\ X & \xrightarrow[j^X]{} & X \cup Y \end{array}$$

in **Top**. *Then there is a long exact sequence*

$$\cdots \to \tilde{H}_n(X \cap Y) \xrightarrow{(i_*^X, i_*^Y)} \tilde{H}_n(X) \oplus \tilde{H}_n(Y) \xrightarrow{j_*^X - j_*^Y} \tilde{H}_n(X \cup Y)$$
$$\xrightarrow{\partial} \tilde{H}_{n-1}(X \cap Y) \to \cdots \to \tilde{H}_0(X \cup Y) \to 0$$

of abelian groups.

Exercise 2. Let $k \geq 0$ and consider the following pushout diagram of inclusions

$$\begin{array}{ccc} S^k & \longrightarrow & D^{k+1} \\ \downarrow & & \downarrow \\ D^{k+1} & \longrightarrow & S^{k+1}. \end{array}$$

in **Top**. Use Theorem 1 and Exercise 1(f) to prove the following.

$$\tilde{H}_n(S^k) \cong \begin{cases} \mathbb{Z}, & n = k, \\ 0, & n \neq k \end{cases} \tag{1}$$

Theorem 2 (Brouwer's fixed point theorem)**.** *Let $n \geq 0$. Every continuous map $f : D^{n+1} \longrightarrow D^{n+1}$ has a fixed point; i.e., there exists a point $x \in D^{n+1}$ such that $f(x) = x$.*

The purpose of the following exercise is to prove Theorem 2 using calculation (1) of the reduced homology of spheres.

Exercise 3. Let $n \geq 0$ and suppose $f : D^{n+1} \longrightarrow D^{n+1}$ is a continuous map. We want to show that f has a fixed point. Suppose not; then $f(x) - x \neq 0$ for every $x \in D^{n+1}$.

(a) Define a function $r : D^{n+1} \longrightarrow S^n$ by setting $r(x)$ to be the point on S^n obtained from the intersection of the line segment from $f(x)$ to x extended to meet S^n; it is useful to draw a picture.
(b) Prove that r is a continuous function.
(c) Verify that r makes the following diagram

$$\begin{array}{ccc} S^n & \xrightarrow{\text{id}} & S^n \\ \subset \downarrow & \nearrow_{r} & \\ D^{n+1} & & \end{array} \tag{2}$$

commute.
(d) Apply the reduced homology functor $\tilde{H}_n : \mathbf{Top}_* \longrightarrow \mathbf{Ab}$ to diagram (2) to obtain a (left-hand side) commutative diagram of the form

$$\begin{array}{ccc} \tilde{H}_n(S^n) & \xrightarrow{\text{id}} & \tilde{H}_n(S^n) \\ \downarrow & \nearrow & \\ \tilde{H}_n(D^{n+1}) & & \end{array} \qquad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\ \downarrow & \nearrow & \\ 0 & & \end{array} \tag{3}$$

in **Ab**.
(e) Conclude that the left-hand diagram in (3) has the form of the right-hand diagram in (3).
(f) This implies that $n = 0$ for every $n \in \mathbb{Z}$, which gives a contradiction. Therefore f must have a fixed point. This completes the proof of Theorem 2.

Exercise 4. Prove the following.

(a) If $n \neq m$, then S^n and S^m are not of the same homotopy type.
(b) If $n \neq m$, then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.

Let X and Y be based spaces and recall that their *wedge* (or one point union), denoted $X \vee Y$, is defined by the pushout diagram of inclusions

$$\begin{array}{ccc} * & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \vee Y \end{array}$$

in **Top**.

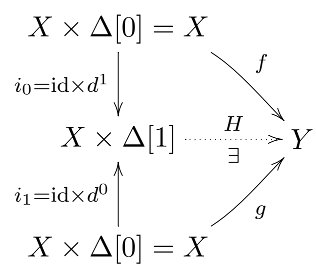
Proposition 3. *Let X, Y be based spaces and $n \geq 0$. Then there are isomorphisms*

$$\tilde{H}_n(X \vee Y) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$$

natural in X, Y.

Exercise 5. Prove Proposition 3.

Let $f, g : X \longrightarrow Y$ be maps of simplicial sets. Recall from lecture that f is *homotopic* to g, denoted $f \simeq g$, if there exists a map H of simplicial sets which makes the diagram



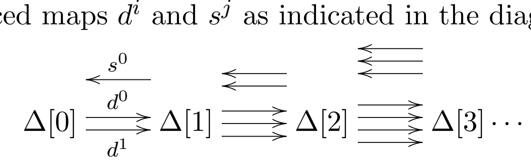
in **sSet** commute. The map H is called a *simplicial homotopy.*

It will be useful to recall from lecture that if Y is a Kan complex, then the simplicial homotopy relation $\simeq$ defines an equivalence relation on $\mathbf{sSet}(X, Y)$, for every simplicial set X.

Exercise 6. Let $f, g : X \longrightarrow Y$ and $f', g' : Y \longrightarrow Z$ be maps of simplicial sets such that $f \simeq g$ and $f' \simeq g'$. Prove the following.

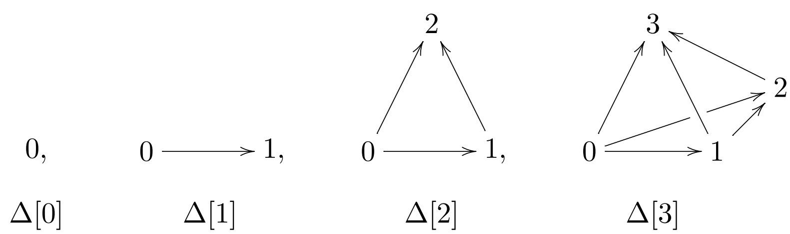
(a) $f'f \simeq f'g$
(b) $f'g \simeq g'g$
(c) Conclude that if Z is a Kan complex, then $f'f \simeq g'g$.

Recall the induced maps d^i and s^j as indicated in the diagram

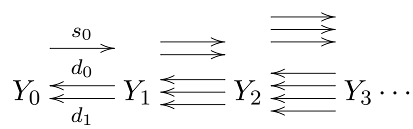


in **sSet**. The following pictures suggest the non-degenerate n-simplices ($n = 0, 1, 2, 3$) of the simplicial sets $\Delta[0], \Delta[1], \Delta[2], \Delta[3]$

(4)



In Exercise 7, it will be helpful to recall [Series 5, Exercise 5] that $\Delta[n] \cong N([n])$; in other words, $\Delta[n]$ is isomorphic to the nerve of the category $[n]$. Also recall that a simplicial set Y has induced maps d_i and s_j as indicated in the diagram



in **Set**. The purpose of the following exercise is to observe that not every simplicial set is a Kan complex.

Exercise 7.

(a) Let Y be a simplicial set and suppose $a, b : \Delta[0] \longrightarrow Y$ are maps of simplicial sets. Prove the following: $a \simeq b$ if and only if there exists a 1-simplex h of Y such that $d_1 h = a$ and $d_0 h = b$.

(b) Consider the maps $d^1, d^0 : \Delta[0] \longrightarrow \Delta[1]$ and note that d^1 (resp. d^0) corresponds to the 0-simplex labeled 0 (resp. 1) in picture (4). Use part (a) to prove that $d^1 \simeq d^0$.

(c) Use part (a) to prove that $d^0 \not\simeq d^1$.

(d) Conclude that $\Delta[1]$ is not a Kan complex.

(e) Let $a, b : \Delta[0] \longrightarrow \Delta[2]$ denote the maps which correspond to the 0-simplices labeled 0 and 2, respectively, in picture (4). Use part (a) to prove that $a \simeq b$ and $b \not\simeq a$.

(f) More generally, let $n \geq 1$ and denote by $a, b : \Delta[0] \longrightarrow \Delta[n]$ the maps which correspond to the 0-simplices labeled 0 and n, respectively, as suggested by picture (4). Use part (a) to prove that $a \simeq b$ and $b \not\simeq a$.

(g) Conclude that $\Delta[n]$ is not a Kan complex for $n \geq 1$.

(h) Observe that $\Delta[0]$ is a Kan complex.

Recall from [Series 5] the following natural decomposition of a simplicial set X in terms of its non-degenerate simplices. This decomposition describes exactly what simplicial sets look like, and will be used below for proving an important property of the geometric realization $|X|$ of X.

Proposition 4. *Let X be a simplicial set. For each $k \geq 0$, denote by $\mathcal{N}X_k \subset X_k$ the set of non-degenerate k-simplices of X. There is a natural isomorphism Ψ of simplicial sets defined levelwise by*

$$\Psi_n : \coprod_{\substack{[n] \twoheadrightarrow [k] \\ \text{in } \Delta}} \mathcal{N}X_k \xrightarrow{\cong} X_n. \tag{5}$$

Here, the coproduct is indexed over the set of all surjections in Δ of the form $\xi : [n] \longrightarrow [k]$, and Ψ_n is the natural map induced by the corresponding maps $\mathcal{N}X_k \subset X_k \xrightarrow{\xi^*} X_n$.

In other words, every simplicial set X is naturally isomorphic to a simplicial set of the form

$$\mathcal{N}X_0 \;\substack{\longrightarrow \\ \longleftarrow \\ \longleftarrow}\; \mathcal{N}X_0 \amalg \mathcal{N}X_1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow}\; \mathcal{N}X_0 \amalg \mathcal{N}X_1 \amalg \mathcal{N}X_1 \amalg \mathcal{N}X_2 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow}\; \cdots$$

constructed entirely from the non-degenerate simplices of X. This leads to the following definition.

Definition 5. Let X be a simplicial set and $m \geq 0$. The *m-skeleton* of X, denoted $\operatorname{Sk}^m X$, is the subcomplex $\operatorname{Sk}^m X \subset X$ which is generated by the k-simplices of X of degree $k \leq m$; i.e.,

$$(\operatorname{Sk}^m X)_n \subset X_n$$

is the set of all $x \in X_n$ such that there exists a surjection $\xi : [n] \longrightarrow [k]$ in Δ, $k \leq m$, and a k-simplex y such that $x = X(\xi)(y)$. Define $\operatorname{Sk}^{-1} X := \emptyset$.

In particular, for every simplicial set X, there is a sequence of inclusions

$$\emptyset = \operatorname{Sk}^{-1} X \subset \operatorname{Sk}^0 X \subset \operatorname{Sk}^1 X \subset \operatorname{Sk}^2 X \subset \cdots \subset \bigcup_{m \geq 0} \operatorname{Sk}^m X = X.$$

In other words, $X \cong \operatorname{colim}_m \operatorname{Sk}^m X$. Note that the 0-skeleton of X is the constant Δ^{op}-shaped diagram with value X_0.

Exercise 8. Prove the following; it is useful to consider the pictures in (4).

(a) $\operatorname{Sk}^0 \Delta[0] = \Delta[0]$.
(b) $\operatorname{Sk}^1 \Delta[1] = \Delta[1]$, $\quad \operatorname{Sk}^0 \Delta[1] = \partial\Delta[1]$.
(c) $\operatorname{Sk}^2 \Delta[2] = \Delta[2]$, $\quad \operatorname{Sk}^1 \Delta[2] = \partial\Delta[2]$, $\quad \operatorname{Sk}^0 \Delta[2] \cong (\Delta[0])^{\amalg 3}$.
(d) $\operatorname{Sk}^p \Delta[m] = \Delta[m]$, $\quad p \geq m$ and $m \geq 0$.
(e) $\operatorname{Sk}^{m-1} \Delta[m] = \partial\Delta[m]$, $\quad m \geq 0$.

The following proposition gives a particularly simple description of the m-skeleton of a simplicial set X in terms of the natural decomposition appearing in Proposition 4; it shows that the subcomplex $\operatorname{Sk}^m X \subset X$ is obtained from the natural decomposition (5) of X by removing all copies of $\mathcal{N}X_k$ with $k > m$.

Proposition 6. *Let X be a simplicial set and $m \geq 0$. For each $k \geq 0$, denote by $\mathcal{N}X_k \subset X_k$ the set of non-degenerate k-simplices of X. There is a natural isomorphism Ψ of simplicial sets defined levelwise by*

$$\Psi_n : \coprod_{\substack{[n]\twoheadrightarrow[k] \\ \text{in } \Delta,\ k\leq m}} \mathcal{N}X_k \xrightarrow{\cong} (\operatorname{Sk}^m X)_n.$$

Here, the coproduct is indexed over the set of all surjections in Δ of the form $\xi : [n]{\longrightarrow}[k]$, $k \leq m$.

Exercise 9. Use Proposition 4 to prove Proposition 6.

The following useful proposition follows easily from Proposition 6.

Proposition 7. *Let X be a simplicial set, $m \geq 0$, and denote by $\mathcal{N}X_m \subset X_m$ the set of non-degenerate m-simplices of X. There are pushout diagrams of the form*

$$\begin{array}{ccc}
\coprod_{x\in\mathcal{N}X_m} \partial\Delta[m] & \longrightarrow & \operatorname{Sk}^{m-1} X \\
\subset\downarrow & & \subset\downarrow \\
\coprod_{x\in\mathcal{N}X_m} \Delta[m] & \longrightarrow & \operatorname{Sk}^{m} X
\end{array}$$

in **sSet**. *In other words, $\operatorname{Sk}^m X$ is obtained from $\operatorname{Sk}^{m-1} X$ by gluing on a set of standard simplices $\Delta[m]$ along their boundaries $\partial\Delta[m]$.*

Exercise 10. Prove Proposition 8 below.

The following proposition shows that the geometric realization $|X|$ of a simplicial set X has the structure of a CW complex. In particular, $|X|$ is a compactly generated Hausdorff space.

Proposition 8. *Let X be a simplicial set, $m \geq 0$, and denote by $\mathcal{N}X_m \subset X_m$ the set of non-degenerate m-simplices of X.*

(a) *There are pushout diagrams of the form*

$$\begin{array}{ccc}
\coprod_{x\in\mathcal{N}X_m} \partial\Delta^m & \longrightarrow & |\operatorname{Sk}^{m-1} X| \\
\subset\downarrow & & \subset\downarrow \\
\coprod_{x\in\mathcal{N}X_m} \Delta^m & \longrightarrow & |\operatorname{Sk}^{m} X|
\end{array}$$

in **Top**.

(b) *There is a sequence of closed inclusions*

$$\emptyset = |\operatorname{Sk}^{-1} X| \subset |\operatorname{Sk}^{0} X| \subset |\operatorname{Sk}^{1} X| \subset |\operatorname{Sk}^{2} X| \subset \cdots \subset \bigcup_{m\geq 0} |\operatorname{Sk}^{m} X| = |X|.$$

In particular, $|X| \cong \operatorname{colim}_m |\operatorname{Sk}^m X|$.

Here are some references for this material: [1, Chapter II], [2, Chapter I] [3, Chapter 2], [4, Chapter 4]

References

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