

Series 8

Let Z be a non-empty topological space, consider the natural map $p : Z \rightarrow *$, and note that $C(*) \cong \mathbb{Z}$; here \mathbb{Z} denotes the chain complex concentrated at 0 with value \mathbb{Z} . Hence $C(Z)$ is an augmented chain complex. The *reduced chain complex* $\tilde{C}(Z)$ of the augmented chain complex $C(Z)$ is defined by the short exact sequence

$$0 \rightarrow \tilde{C}(Z) \rightarrow C(Z) \xrightarrow{p_*} C(*) \rightarrow 0$$

of chain complexes. The *reduced homology* of Z , denoted $\tilde{H}(Z)$, is defined by $\tilde{H}_n(Z) := H_n(\tilde{C}(Z))$.

Exercise 1. Let Z be a non-empty topological space. Prove the following.

- (a) $H_n(Z) \cong \tilde{H}_n(Z)$, for all $n \geq 1$.
- (b) There is a short exact sequence

$$0 \rightarrow \tilde{H}_0(Z) \rightarrow H_0(Z) \rightarrow \mathbb{Z} \rightarrow 0$$

of abelian groups.

- (c) $H_0(Z) \cong \tilde{H}_0(Z) \oplus \mathbb{Z}$.
- (d) $\tilde{H}_n(S^0) = 0$, for all $n \geq 1$.
- (e) $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- (f) $\tilde{H}_0(S^0) \cong \mathbb{Z}$.

The following is the reduced version of the Mayer-Vietoris sequence.

Theorem 1 (Reduced Mayer-Vietoris Theorem). *Let Z be a topological space and suppose $X, Y \subset Z$ are subspaces such that $Z = \text{int}(X) \cup \text{int}(Y)$. Assume $X \cap Y \neq \emptyset$. Consider the commutative diagram of inclusions*

$$\begin{array}{ccc} X \cap Y & \xrightarrow{i^Y} & Y \\ i^X \downarrow & & \downarrow j^Y \\ X & \xrightarrow{j^X} & X \cup Y \end{array}$$

in **Top**. Then there is a long exact sequence

$$\begin{aligned} \cdots \rightarrow \tilde{H}_n(X \cap Y) &\xrightarrow{(i_*^X, i_*^Y)} \tilde{H}_n(X) \oplus \tilde{H}_n(Y) \xrightarrow{j_*^X - j_*^Y} \tilde{H}_n(X \cup Y) \\ &\xrightarrow{\partial} \tilde{H}_{n-1}(X \cap Y) \rightarrow \cdots \rightarrow \tilde{H}_0(X \cup Y) \rightarrow 0 \end{aligned}$$

of abelian groups.

Exercise 2. Let $k \geq 0$ and consider the following pushout diagram of inclusions

$$\begin{array}{ccc} S^k & \longrightarrow & D^{k+1} \\ \downarrow & & \downarrow \\ D^{k+1} & \longrightarrow & S^{k+1}. \end{array}$$

in **Top**. Use Theorem 1 and Exercise 1(f) to prove the following.

$$(1) \quad \tilde{H}_n(S^k) \cong \begin{cases} \mathbb{Z}, & n = k, \\ 0, & n \neq k \end{cases}$$

Theorem 2 (Brouwer's fixed point theorem). *Let $n \geq 0$. Every continuous map $f : D^{n+1} \rightarrow D^{n+1}$ has a fixed point; i.e., there exists a point $x \in D^{n+1}$ such that $f(x) = x$.*

The purpose of the following exercise is to prove Theorem 2 using calculation (1) of the reduced homology of spheres.

Exercise 3. Let $n \geq 0$ and suppose $f : D^{n+1} \rightarrow D^{n+1}$ is a continuous map. We want to show that f has a fixed point. Suppose not; then $f(x) - x \neq 0$ for every $x \in D^{n+1}$.

- (a) Define a function $r : D^{n+1} \rightarrow S^n$ by setting $r(x)$ to be the point on S^n obtained from the intersection of the line segment from $f(x)$ to x extended to meet S^n ; it is useful to draw a picture.
- (b) Prove that r is a continuous function.
- (c) Verify that r makes the following diagram

$$(2) \quad \begin{array}{ccc} S^n & \xrightarrow{\text{id}} & S^n \\ \downarrow \subset & \nearrow r & \\ D^{n+1} & & \end{array}$$

commute.

- (d) Apply the reduced homology functor $\tilde{H}_n : \mathbf{Top}_* \rightarrow \mathbf{Ab}$ to diagram (2) to obtain a (left-hand side) commutative diagram of the form

$$(3) \quad \begin{array}{ccc} \tilde{H}_n(S^n) & \xrightarrow{\text{id}} & \tilde{H}_n(S^n) \\ \downarrow & \nearrow & \\ \tilde{H}_n(D^{n+1}) & & \end{array} \quad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\ \downarrow & \nearrow & \\ 0 & & \end{array}$$

in **Ab**.

- (e) Conclude that the left-hand diagram in (3) has the form of the right-hand diagram in (3).
- (f) This implies that $n = 0$ for every $n \in \mathbb{Z}$, which gives a contradiction. Therefore f must have a fixed point. This completes the proof of Theorem 2.

Exercise 4. Prove the following.

- (a) If $n \neq m$, then S^n and S^m are not of the same homotopy type.
- (b) If $n \neq m$, then \mathbb{R}^n and \mathbb{R}^m are not homeomorphic.

Let X and Y be based spaces and recall that their *wedge* (or one point union), denoted $X \vee Y$, is defined by the pushout diagram of inclusions

$$\begin{array}{ccc} * & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \vee Y \end{array}$$

in **Top**.

Proposition 3. Let X, Y be based spaces and $n \geq 0$. Then there are isomorphisms

$$\tilde{H}_n(X \vee Y) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$$

natural in X, Y .

Exercise 5. Prove Proposition 3.

Let $f, g : X \rightarrow Y$ be maps of simplicial sets. Recall from lecture that f is *homotopic* to g , denoted $f \simeq g$, if there exists a map H of simplicial sets which makes the diagram

$$\begin{array}{ccc} X \times \Delta[0] = X & & \\ \downarrow i_0 = \text{id} \times d^1 & \searrow f & \\ X \times \Delta[1] & \xrightarrow{H} & Y \\ \uparrow i_1 = \text{id} \times d^0 & \swarrow g & \\ X \times \Delta[0] = X & & \end{array}$$

\exists

in **sSet** commute. The map H is called a *simplicial homotopy*.

It will be useful to recall from lecture that if Y is a Kan complex, then the simplicial homotopy relation \simeq defines an equivalence relation on $\mathbf{sSet}(X, Y)$, for every simplicial set X .

Exercise 6. Let $f, g : X \rightarrow Y$ and $f', g' : Y \rightarrow Z$ be maps of simplicial sets such that $f \simeq g$ and $f' \simeq g'$. Prove the following.

- (a) $f'f \simeq f'g$
- (b) $f'g \simeq g'g$
- (c) Conclude that if Z is a Kan complex, then $f'f \simeq g'g$.

Recall the induced maps d^i and s^j as indicated in the diagram

$$\begin{array}{ccccccc} & \xleftarrow{s^0} & & \xleftarrow{\quad} & \xleftarrow{\quad} & \xleftarrow{\quad} & \\ \Delta[0] & \xrightarrow{d^0} & \Delta[1] & \xrightarrow{\quad} & \Delta[2] & \xrightarrow{\quad} & \Delta[3] \cdots \\ & \xrightarrow{d^1} & & \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} & \end{array}$$

Proposition 4. *Let X be a simplicial set. For each $k \geq 0$, denote by $\mathcal{N}X_k \subset X_k$ the set of non-degenerate k -simplices of X . There is a natural isomorphism Ψ of simplicial sets defined levelwise by*

$$(5) \quad \Psi_n : \coprod_{\substack{[n] \rightarrow [k] \\ \text{in } \Delta}} \mathcal{N}X_k \xrightarrow{\cong} X_n.$$

Here, the coproduct is indexed over the set of all surjections in Δ of the form $\xi : [n] \rightarrow [k]$, and Ψ_n is the natural map induced by the corresponding maps $\mathcal{N}X_k \subset X_k \xrightarrow{\xi^*} X_n$.

In other words, every simplicial set X is naturally isomorphic to a simplicial set of the form

$$\begin{array}{ccccccc} & \longrightarrow & \rightrightarrows & \rightrightarrows & \rightrightarrows & \rightrightarrows & \rightrightarrows \\ \mathcal{N}X_0 & \xleftarrow{\quad} & \mathcal{N}X_0 \amalg \mathcal{N}X_1 & \xleftarrow{\quad} & \mathcal{N}X_0 \amalg \mathcal{N}X_1 \amalg \mathcal{N}X_1 & \xleftarrow{\quad} & \mathcal{N}X_0 \amalg \mathcal{N}X_1 \amalg \mathcal{N}X_2 & \xleftarrow{\quad} \cdots \end{array}$$

constructed entirely from the non-degenerate simplices of X . This leads to the following definition.

Definition 5. Let X be a simplicial set and $m \geq 0$. The m -skeleton of X , denoted $\text{Sk}^m X$, is the subcomplex $\text{Sk}^m X \subset X$ which is generated by the k -simplices of X of degree $k \leq m$; i.e.,

$$(\text{Sk}^m X)_n \subset X_n$$

is the set of all $x \in X_n$ such that there exists a surjection $\xi : [n] \rightarrow [k]$ in Δ , $k \leq m$, and a k -simplex y such that $x = X(\xi)(y)$. Define $\text{Sk}^{-1} X := \emptyset$.

In particular, for every simplicial set X , there is a sequence of inclusions

$$\emptyset = \text{Sk}^{-1} X \subset \text{Sk}^0 X \subset \text{Sk}^1 X \subset \text{Sk}^2 X \subset \cdots \subset \bigcup_{m \geq 0} \text{Sk}^m X = X.$$

In other words, $X \cong \text{colim}_m \text{Sk}^m X$. Note that the 0-skeleton of X is the constant Δ^{op} -shaped diagram with value X_0 .

Exercise 8. Prove the following; it is useful to consider the pictures in (4).

- (a) $\text{Sk}^0 \Delta[0] = \Delta[0]$.
- (b) $\text{Sk}^1 \Delta[1] = \Delta[1]$, $\text{Sk}^0 \Delta[1] = \partial \Delta[1]$.
- (c) $\text{Sk}^2 \Delta[2] = \Delta[2]$, $\text{Sk}^1 \Delta[2] = \partial \Delta[2]$, $\text{Sk}^0 \Delta[2] \cong (\Delta[0])^{\amalg 3}$.
- (d) $\text{Sk}^p \Delta[m] = \Delta[m]$, $p \geq m$ and $m \geq 0$.
- (e) $\text{Sk}^{m-1} \Delta[m] = \partial \Delta[m]$, $m \geq 0$.

The following proposition gives a particularly simple description of the m -skeleton of a simplicial set X in terms of the natural decomposition appearing in Proposition 4; it shows that the subcomplex $\text{Sk}^m X \subset X$ is obtained from the natural decomposition (5) of X by removing all copies of $\mathcal{N}X_k$ with $k > m$.

Proposition 6. *Let X be a simplicial set and $m \geq 0$. For each $k \geq 0$, denote by $\mathcal{N}X_k \subset X_k$ the set of non-degenerate k -simplices of X . There is a natural isomorphism Ψ of simplicial sets defined levelwise by*

$$\Psi_n : \coprod_{\substack{[n] \rightarrow [k] \\ \text{in } \Delta, k \leq m}} \mathcal{N}X_k \xrightarrow{\cong} (\text{Sk}^m X)_n.$$

Here, the coproduct is indexed over the set of all surjections in Δ of the form $\xi : [n] \rightarrow [k]$, $k \leq m$.

Exercise 9. Use Proposition 4 to prove Proposition 6.

The following useful proposition follows easily from Proposition 6.

Proposition 7. *Let X be a simplicial set, $m \geq 0$, and denote by $\mathcal{N}X_m \subset X_m$ the set of non-degenerate m -simplices of X . There are pushout diagrams of the form*

$$\begin{array}{ccc} \coprod_{x \in \mathcal{N}X_m} \partial\Delta[m] & \longrightarrow & \text{Sk}^{m-1} X \\ \subset \downarrow & & \subset \downarrow \\ \coprod_{x \in \mathcal{N}X_m} \Delta[m] & \longrightarrow & \text{Sk}^m X \end{array}$$

in \mathbf{sSet} . In other words, $\text{Sk}^m X$ is obtained from $\text{Sk}^{m-1} X$ by gluing on a set of standard simplices $\Delta[m]$ along their boundaries $\partial\Delta[m]$.

Exercise 10. Prove Proposition 8 below.

The following proposition shows that the geometric realization $|X|$ of a simplicial set X has the structure of a CW complex. In particular, $|X|$ is a compactly generated Hausdorff space.

Proposition 8. *Let X be a simplicial set, $m \geq 0$, and denote by $\mathcal{N}X_m \subset X_m$ the set of non-degenerate m -simplices of X .*

(a) *There are pushout diagrams of the form*

$$\begin{array}{ccc} \coprod_{x \in \mathcal{N}X_m} \partial\Delta^m & \longrightarrow & |\text{Sk}^{m-1} X| \\ \subset \downarrow & & \subset \downarrow \\ \coprod_{x \in \mathcal{N}X_m} \Delta^m & \longrightarrow & |\text{Sk}^m X| \end{array}$$

in \mathbf{Top} .

(b) *There is a sequence of closed inclusions*

$$\emptyset = |\text{Sk}^{-1} X| \subset |\text{Sk}^0 X| \subset |\text{Sk}^1 X| \subset |\text{Sk}^2 X| \subset \cdots \subset \bigcup_{m \geq 0} |\text{Sk}^m X| = |X|.$$

In particular, $|X| \cong \text{colim}_m |\text{Sk}^m X|$.

Here are some references for this material: [1, Chapter II], [2, Chapter I] [3, Chapter 2], [4, Chapter 4]

REFERENCES

- [1] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.
- [2] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.
- [3] Brayton Gray. *Homotopy theory*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975. An introduction to algebraic topology, Pure and Applied Mathematics, Vol. 64.
- [4] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York, 1981. Corrected reprint.