Homology and cohomology **Spring 2009**
Prof. Kathryn Hess

Series 9

The following is a useful characterization of split short exact sequences.

Proposition 1. *Consider any short exact sequence* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *of abelian groups. The following properties are equivalent:*

(i) *β has a right inverse* $\beta' : C \longrightarrow B$ *with* $\beta\beta' = \mathrm{id}$.
(ii) *α has a left inverse* $\alpha' : B \longrightarrow A$ *with* $\alpha'\alpha = \mathrm{id}$.
(iii) *There is an isomorphism θ which makes the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \| & & \cong \uparrow \theta & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & A \oplus C & \xrightarrow{p} & C & \longrightarrow & 0 \end{array}$$

of abelian groups commute; here, the maps i and p are defined by $i(a) := (a, 0)$ *and* $p(a, c) = c$*, respectively.*

Exercise 1. The purpose of this exercise is to prove Proposition 1.

(a) Prove that (iii) implies (i).
(b) Prove that (iii) implies (ii).
(c) Prove that (i) implies the following: there exists an isomorphism ϕ which makes the diagram

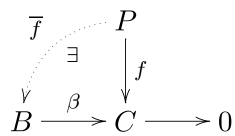
$$\begin{array}{ccccc} B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ \cong \uparrow \phi & & \| & & \\ (\ker \beta) \oplus C & \xrightarrow{p} & C & \longrightarrow & 0 \end{array}$$

commute; here, the map p is defined by $p(b, c) := c$.
(d) Prove that (ii) implies the following: there exists an isomorphism ψ which makes the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B \\ & & \| & & \cong \downarrow \psi \\ 0 & \longrightarrow & A & \xrightarrow{i} & A \oplus (B/\alpha A) \end{array}$$

commute; here, the map i is defined by $i(a) := (a, 0)$.
(e) Use part (c) to prove that (i) implies (iii).
(f) Use part (d) to prove that (ii) implies (iii).

Definition 2. An abelian group P is *projective* if for each solid diagram



of abelian groups with β an epimorphism, there exists a homomorphism $\overline{f}$ which makes the diagram commute.

Proposition 3. *Every free abelian group is projective.*

Exercise 2. Prove Proposition 3.

To get started on Exercise 2, try using the universal property of the free abelian group generated by a set.

Proposition 4. *An abelian group P is projective if and only if it is a direct summand of a free abelian group; i.e., if and only if there is an isomorphism*

$$P \oplus Q \cong F$$

for some abelian group Q and free abelian group F.

Exercise 3. Prove Proposition 4.

To get started on Exercise 3, consider the free abelian group

$$F := \bigoplus_{p \in P} \mathbb{Z}$$

generated by the underlying set of P, together with the natural epimorphism $F \to P$.

Remark 5. Since every subgroup of a free abelian group is a free abelian group, it follows from Proposition 4 that every projective abelian group is a free abelian group.

Proposition 6. *The following properties of an abelian group P are equivalent:*

(i) *P is projective.*

(ii) *Every short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} P \to 0$ of abelian groups splits.*

Exercise 4. Prove Proposition 6.

Definition 7. An abelian group G is *flat* if the functor $-\otimes G$ preserves monomorphisms; i.e., if for any monomorphism $\alpha : A \longrightarrow B$ of abelian groups, the induced map $i \otimes \mathrm{id} : A \otimes G \longrightarrow B \otimes G$ is a monomorphism.

Proposition 8. *Every projective abelian group is flat.*

Remark 9. Not every flat abelian group is projective. For example, $G := \prod_{n \geq 1} \mathbb{Z}$ is a flat abelian group which is not projective.

Exercise 5. The purpose of this exercise is to prove Proposition 8.

(a) Prove that $\mathbb{Z}$ is flat.
(b) Use part (a) to prove that every free abelian group is flat.
(c) Use part (b) to prove that every projective abelian group is flat.

Recall from [Series 1] the following proposition.

Proposition 10. *If G is an abelian group and $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is an exact sequence of abelian groups, then*

$$A \otimes G \xrightarrow{\alpha\otimes \mathrm{id}} B \otimes G \xrightarrow{\beta\otimes \mathrm{id}} C \otimes G \to 0$$

is an exact sequence (of abelian groups).

Proposition 11. *The following properties of an abelian group G are equivalent:*

(a) *G is flat.*
(b) *The functor $-\otimes G$ preserves short exact sequences of abelian groups.*

Exercise 6. Use Proposition 10 to prove Proposition 11.

Let G be an abelian group. Recall from lecture that a *resolution* X of G is an exact sequence

$$\cdots \to X_n \xrightarrow{\partial} X_{n-1} \xrightarrow{\partial} \cdots \to X_1 \xrightarrow{\partial} X_0 \xrightarrow{\varepsilon} G \to 0$$

of abelian groups. The chain complex X, which has the form

$$\cdots \to X_n \xrightarrow{\partial} X_{n-1} \xrightarrow{\partial} \cdots \to X_1 \xrightarrow{\partial} X_0 \to 0,$$

is *free* (resp. *projective*) if each X_n is free (resp. projective). Also recall from lecture that every abelian group G has a free resolution F of the form

$$0 \to F_1 \to F_0 \to G \to 0. \tag{1}$$

We know by Proposition 10 that the functor $-\otimes G$ is right exact. The following proposition indicates how the $\mathrm{Tor}(-, G)$ functor provides a measure of the inexactitude of $-\otimes G$.

Proposition 12. *Let G be an abelian group and let F be a free resolution of G of the form* (1). *Consider any short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of abelian groups. Then there is a short exact sequence*

$$0 \to A \otimes F \xrightarrow{\alpha\otimes \mathrm{id}} B \otimes F \xrightarrow{\beta\otimes \mathrm{id}} C \otimes F \to 0$$

of chain complexes, and hence a corresponding long exact sequence

$$\begin{aligned} 0 \to \mathrm{Tor}(A, G) \to \mathrm{Tor}(B, G) \to \mathrm{Tor}(C, G) \\ \xrightarrow{\partial} A \otimes G \to B \otimes G \to C \otimes G \to 0 \end{aligned}$$

of abelian groups.

Exercise 7. Use Proposition 11 to prove Proposition 12.

Exercise 8. Let A, B, G be abelian groups. Let $\{A_t\}_t$ be a collection of abelian groups indexed on a set. Define $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ for each $n \geq 1$. Prove the following.

(a) $\mathrm{Tor}(A, P) = 0$, for all projective P.
(b) $\mathrm{Tor}(A \oplus B, G) \cong \mathrm{Tor}(A, G) \oplus \mathrm{Tor}(B, G)$.
(c) $\mathrm{Tor}(\oplus_t A_t, G) \cong \oplus_t \mathrm{Tor}(A_t, G)$.
(d) $\mathrm{Tor}(A, G) \cong \mathrm{Tor}(G, A)$.
(e) $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong Z_d$, with $d := \gcd(m, n)$.

To get started on part (e), consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}_m \to 0$$

of abelian groups together with Proposition 12.

Exercise 9. Let $f, g : X \longrightarrow Y$ be maps of chain complexes. Suppose there exists a map h of chain complexes which makes the diagram

$$\begin{array}{ccc} X\otimes C\Delta[0] = X & & \\ {\scriptstyle i_0=\mathrm{id}\otimes d^1}\downarrow & \searrow^{f} & \\ X\otimes C\Delta[1] & \overset{h}{\underset{\exists}{\dashrightarrow}} & Y \\ {\scriptstyle i_1=\mathrm{id}\otimes d^0}\uparrow & \nearrow_{g} & \\ X\otimes C\Delta[0] = X & & \end{array}$$

commute. Prove that f and g are chain homotopic.

Here are some references for this material: [1, Chapters I and V]

References

[1] S. Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.