Homotopie et Homologie
Exercise Set 1

22.09.2011

1. Show that if $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ are functors, then the pair of functions

$$G_0 \circ F_0 : \operatorname{Ob}\mathscr{C} \to \operatorname{Ob}\mathscr{E} \quad \text{and} \quad G_1 \circ F_1 : \operatorname{Mor}\mathscr{C} \to \operatorname{Mor}\mathscr{E}$$

defines a functor $G \circ F : \mathscr{C} \to \mathscr{E}$.

2. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Show that if we set

$$\operatorname{Ob}(\mathscr{C} \times \mathscr{D}) = \operatorname{Ob}\mathscr{C} \times \operatorname{Ob}\mathscr{D}$$

and for all $C, C' \in \operatorname{Ob}\mathscr{C}$, $D, D' \in \operatorname{Ob}\mathscr{D}$

$$(\mathscr{C} \times \mathscr{D})\big((C,D),(C',D')\big) = \mathscr{C}(C,C') \times \mathscr{D}(D,D'),$$

then there is a composition law making $\mathscr{C} \times \mathscr{D}$ into a category such that projection onto the first (respectively, second) component induces functors $P_1 : \mathscr{C} \times \mathscr{D} \to \mathscr{C}$ and $P_2 : \mathscr{C} \times \mathscr{D} \to \mathscr{D}$. Show moreover that for every pair of functors $F : \mathscr{E} \to \mathscr{C}$ and $G : \mathscr{E} \to \mathscr{D}$, there is a unique functor $(F,G) : \mathscr{E} \to \mathscr{C} \times \mathscr{D}$ such that $P_1 \circ (F,G) = F$ and $P_2 \circ (F,G) = G$.

3. A morphism $f : A \to B$ in a category $\mathscr{C}$ is an *isomorphism* if there exists another morphism $g : B \to A$ in $\mathscr{C}$ such that $g \circ f = \mathrm{Id}_A$ and $f \circ g = \mathrm{Id}_B$.

   (a) What are the isomorphisms in **Set**, **Top**, and **Gr**? What about in the category **G** arising from a group G?

   (b) Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Show that if $f : A \to B$ is an isomorphism in $\mathscr{C}$, then $F_1(f)$ is an isomorphism in $\mathscr{D}$. Does the converse hold?

4. Let $(X, \mathscr{T})$ be a topological space. Explain how to form a category $\Pi_1(X,\mathscr{T})$ with $\operatorname{Ob}\Pi_1(X,\mathscr{T}) = X$ and such that $\Pi_1(X,\mathscr{T})(x,x')$ is the set of all homotopy classes of paths from x to x'. Show that every morphism in $\Pi_1(X,\mathscr{T})$ is an isomorphism. (This category is called the *fundamental groupoid* of the space $(X,\mathscr{T})$.)