Homotopie et Homologie
Exercise Set 10

01.12.2011

Throughout these exercises, *space* means *topological space*, *map* means *continuous map*, and I denotes $[0, 1]$.

1. Prove that a composite of fibrations is a fibration.

2. Prove that a pullback of a fibration along any map is a fibration.

3. Use the characterization of Hurewicz fibrations to prove that the map

$$(ev_0, ev_1) : \mathrm{Map}(I, X) \to X \times X : \lambda \mapsto \big(\lambda(0), \lambda(1)\big)$$

   is a Hurewicz fibration.

4. Prove that "fibrations are everywhere", i.e., for every continuous map $f : X \to Y$, there exist a homotopy equivalence $j : X \to E$ and a (Hurewicz) fibration $p : E \to Y$ such that $f = p \circ j$.

   Hint 1. Let E be the pullback of $ev_0 : \mathrm{Map}(I, Y) \to Y$ and of $f : X \to Y$.

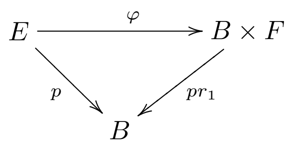
5. Let $p : E \to B$ be a Hurewicz fibration. Prove that if B is path-connected, then for all $b_0, b_1 \in B$,

$$p^{-1}(b_0) \sim p^{-1}(b_1).$$

   Hint 2. Use the path lifting map $\Gamma : E \times_B \mathrm{Map}(I, B) \to \mathrm{Map}(I, E)$ associated to p.

6. Let $p : E \to B$ be a Hurewicz fibration, and let $b_0 \in B$. Let $F = p^{-1}(b_0)$.

   (a) Prove that if $B \simeq \{b_0\}$, then there is a homotopy equivalence $\varphi : E \to B \times F$ such that



   commutes.

Hint 3. Use the path lifting map $\Gamma : E \times_B \mathrm{Map}(I, B) \to \mathrm{Map}(I, E)$ associated to p and the function $\alpha : \mathrm{Map}(B \times I, B) \to \mathrm{Map}\big(B, \mathrm{Map}(I, B)\big)$ from the very first lecture.

Remark 4. For the remainder of the exercise, we no longer assume that B is contractible.

(b) Prove that the homotopy fiber of p (cf. Definition 2, Exercise set 7) is homotopy equivalent to F.

(c) Prove that there is an exact sequence

$$\cdots \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \cdots \to \pi_1 B \to \pi_0 F \to \pi_0 E \to \pi_0 B$$

(where homotopy groups of B are calculated with respect to b_0, while those of E and F are calculated with respect to some $e_0 \in F$) and describe the homomorphisms in the sequence (cf. Exercise 2, Exercise set 8). This is the *long exact sequence in homotopy* of the fibration p.

Remark 5. In the textbook, the authors prove the existence of such long exact sequences for all Serre fibrations as well (Corollary 4.3.34).