Homotopie et Homologie
Exercise Set 11

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Throughout these exercises, *space* means *topological space* and *map* means *continuous map*.

1. Let X be a CW-complex. Prove the following properties of the topology on X.

   (a) The n-skeleton X_n of X is closed in X.

   (b) If X is connected, then X is path-connected.

   (c) X is a normal space.
   Hint 1. Prove by induction on n, using the Tietze extension theorem, that for all closed subsets A and B of X, there exists continuous maps $f_n : X_n \to I$ for all $n \geq 0$ such that $f_n(A \cap X_n) = \{0\}$, $f_n(B \cap X_n) = \{1\}$ and $f_n|_{X_{n-1}} = f_{n-1}$.

   (d) If C is a compact subset of X, then there is some n such that $C \subseteq X_n$.
   Hint 2. If C is not contained in X_n for any n, then there is a set $\{x_n \mid n \in \mathbb{N}\}$ such that $x_n \in C \setminus X_n$ for all n. Show that the sequence

   $$\cdots \subset \{x_n \mid n \geq 2\} \subset \{x_n \mid n \geq 1\} \subset \{x_n \mid n \geq 0\}$$

   violates the Finite Intersection Property for compact sets.

2. Show that if X and Y are CW-complexes, and X and Y both have countably many cells, then $X \times Y$ is also a CW-complex.
   Hint 3. For all $0 \leq k \leq n$,

   $$(D^n, S^{n-1}) \cong (D^k \times D^{n-k}, S^{k-1} \times D^{n-k} \cup D^k \times S^{n-k-1}).$$

3. Let G be a discrete group acting on the right on a space X, i.e., there is a map $\rho : X \times G \to X : (x, a) \mapsto x \cdot a$ such that $(x \cdot a) \cdot b = x \cdot (ab)$ for all $x \in X$ and $a, b \in G$ and $x \cdot e = x$ for all $x \in X$, where e is the neutral element of G. Show that if X is a CW-complex, and ρ is cellular, then the orbit space X/G is a CW-complex.

7. Let $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\} \subset \mathbb{R}$. Show that X is not homotopy equivalent to a CW-complex.