Throughout these exercises, space means topological space and map means continuous map.

1. Let $X$ be a CW-complex. Prove the following properties of the topology on $X$.

(a) The $n$-skeleton $X_n$ of $X$ is closed in $X$.
(b) If $X$ is connected, then $X$ is path-connected.
(c) $X$ is a normal space.

**Hint 1.** Prove by induction on $n$, using the Tietze extension theorem, that for all closed subsets $A$ and $B$ of $X$, there exists continuous maps $f_n : X_n \to I$ for all $n \geq 0$ such that $f_n(A \cap X_n) = \{0\}$, $f_n(B \cap X_n) = \{1\}$ and $f_n|_{X_{n-1}} = f_{n-1}$.

(d) If $C$ is a compact subset of $X$, then there is some $n$ such that $C \subseteq X_n$.

**Hint 2.** If $C$ is not contained in $X_n$ for any $n$, then there is a set $\{x_n \mid n \in \mathbb{N}\}$ such that $x_n \in C \setminus X_n$ for all $n$. Show that the sequence $\cdots \subseteq \{x_n \mid n \geq 2\} \subseteq \{x_n \mid n \geq 1\} \subseteq \{x_n \mid n \geq 0\}$ violates the Finite Intersection Property for compact sets.

2. Show that if $X$ and $Y$ are CW-complexes, and $X$ and $Y$ both have countably many cells, then $X \times Y$ is also a CW-complex.

**Hint 3.** For all $0 \leq k \leq n$,

$$(D^n, S^{n-1}) \cong (D^k \times D^{n-k}, S^{k-1} \times D^{n-k} \cup D^k \times S^{n-k-1}).$$

3. Let $G$ be a discrete group acting on the right on a space $X$, i.e., there is a map $\rho : X \times G \to X : (x, a) \mapsto x \cdot a$ such that $(x \cdot a) \cdot b = x \cdot (ab)$ for all $x \in X$ and $a, b \in G$ and $x \cdot e = x$ for all $x \in X$, where $e$ is the neutral element of $G$. Show that if $X$ is a CW-complex, and $\rho$ is cellular, then the orbit space $X/G$ is a CW-complex.

7. Let $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\} \subset \mathbb{R}$. Show that $X$ is not homotopy equivalent to a CW-complex.