Homotopie et Homologie
Exercise Set 12

15.12.2011

Throughout these exercises, *space* means *topological space* and *map* means *continuous map.*

1. *Complex projective space* of dimenion n is

$$\mathbb{C}P^n = \mathbb{C}^{n+1} \smallsetminus \{0\}/\sim$$

   where $z \sim z'$ if and only if there exists $\lambda \in \mathbb{C}$ such that $z = \lambda z'$. Prove that $SP^n(S^2) \cong \mathbb{C}P^n$.

2. Let (X, x_0) be a pointed CW-complex.

   (a) Show that concatenation of sequences

$$X^m \times X^n \to X^{m+n} : \big((x_1, ..., x_m), (x'_1, ..., x'_n)\big) \mapsto (x_1, ..., x_m, x'_1, ..., x'_n)$$

   induces a commutative binary operation

$$+ : SP(X) \times SP(X) \to SP(X),$$

   which is continuous if, for example, X has countably many cells.

   (b) Let (W, w_0) be a compact, pointed space. Show that $+$ induces a commutative binary operation on $[W, SP(X)]_*$, for any pointed CW-complex (X, x_0).

Definition 1. Let G be an abelian group, and let $n \geq 2$. An *Eilenberg-MacLane space of type* (G, n) is a pointed space (X, x_0) such that $\pi_n(X, x_0) \cong G$ and $\pi_k(X, x_0)) = 0$ if $k \neq n$.

Conventions 2. An Eilenberg-MacLane space of type (G, n) is usually denoted $K(G, n)$.

3. Assuming that the natural map $S^1 \to SP(S^1)$ is a weak equivalence, show that $SP(S^n)$ is an Eilenberg-MacLane space of type $(\mathbb{Z}, n)$ for all $n \geq 1$.

4. Let $m \in \mathbb{N}$, and define $\lambda_m : S^1 \to S^1$ by $\lambda_m(e^{i\theta}) = e^{im\theta}$.
   (a) Prove that $\pi_1\lambda_m : \pi_1 S^1 \to \pi_1 S^1$ is given by multiplication by m.
   (b) Prove that $SP(C_{\lambda_m})$ is an Eilenberg-MacLane space of type $(\mathbb{Z}/m, 1)$.
   Hint 3. Use the quasi-fibration obtained by applying SP to the sequence $S^1 \xrightarrow{\lambda_m} S^1 \to C_{\lambda_m}$.
   (c) Let $n > 1$. Prove that $SP(\Sigma^{n-1}C_{\lambda_m})$ is an Eilenberg-MacLane space of type $(\mathbb{Z}/m, n)$.
   (d) Let G be any finitely generated abelian group, and let $n \geq 1$. Explain how to construct an Eilenberg-MacLane space of type $K(G, n)$.

5. (Bonus, for those who like categories...) In this exercise we show that the infinite symmetric product construction is a *monad* and that abelian topological groups are *Eilenberg-Moore algebras* for this monad.
   Let $\iota_n^X : SP^n(X) \to SP^{n+1}(X)$ denote the canonical inclusion, and let $\eta^X : X \to SP(X)$ denote the inclusion of X as $SP^1(X)$.
   (a) For any CW-complex X and for all $n, m \geq 0$, define maps
   $$\mu^X_{n,m} : SP^n(SP^m(X)) \to SP^{n+m}(X)$$
   such that
   $$\begin{array}{ccccc} SP^{n+1}(SP^m(X)) & \xleftarrow{\iota_n^{SP^m X}} & SP^n(SP^m(X)) & \xrightarrow{SP^n(\iota_m^X)} & SP^n(SP^{m+1}(X)) \\ \downarrow{\scriptstyle \mu^X_{n+1,m}} & & \downarrow{\scriptstyle \mu^X_{n,m}} & & \downarrow{\scriptstyle \mu^X_{n,m+1}} \\ SP^{n+m+1}(X) & \xleftarrow{\iota^X_{n+m}} & SP^{n+m}(X) & \xrightarrow{\iota^X_{n+m}} & SP^{n+m+1}(X) \end{array}$$
   and
   $$\begin{array}{ccc} SP^n(SP^m(X)) & \xrightarrow{\mu^X_{n,m}} & SP^{n+m}(X) \\ \downarrow{\scriptstyle SP^n(SP^m f)} & & \downarrow{\scriptstyle SP^{n+m} f} \\ SP^n(SP^m(Y)) & \xrightarrow{\mu^Y_{n,m}} & SP^{n+m}(Y) \end{array}$$
   commute, where $f : X \to Y$ is any cellular map of CW-complexes.
   (b) From the collection $\{\mu^X_{n,m} \mid m, n \geq 0\}$, construct a map
   $$\mu^X : SP(SP(X)) \to SP(X)$$
   such that
   $$\begin{array}{ccc} SP(SP(X)) & \xrightarrow{\mu^X} & SP(X) \\ \downarrow{\scriptstyle SP(SP f)} & & \downarrow{\scriptstyle SP f} \\ SP(SP(Y)) & \xrightarrow{\mu^Y} & SP(Y) \end{array}$$
   commutes, for any cellular map $f : X \to Y$.

(c) Show that

$$\begin{array}{ccc} SP\big(SP(SP(X))\big) & \xrightarrow{\mu^{SP(X)}} & SP(SP(X)) \\ {\scriptstyle SP(\mu^X)}\downarrow & & \downarrow{\scriptstyle \mu^X} \\ SP(SP(X)) & \xrightarrow{\mu^X} & SP(X) \end{array}$$

and

$$\begin{array}{ccccc} SP(X) & \xrightarrow{\eta^{SP(X)}} & SP(SP(X)) & \xleftarrow{SP\,\eta^X} & SP(X) \\ & {\scriptstyle Id_{SP(X)}}\searrow & \downarrow{\scriptstyle \mu^X} & \swarrow{\scriptstyle Id_{SP(X)}} & \\ & & SP(X) & & \end{array}$$

commute for all CW-complexes X.

(d) Let A be a topological abelian group with the structure of a CW-complex such that the multiplication map is cellular, e.g., S^1 with its usual CW-structure or any abelian group with the discrete topology. Show that there is a map $m : SP(A) \to A$ such that

$$\begin{array}{ccc} SP(SP(A)) & \xrightarrow{\mu^A} & SP(A) \\ {\scriptstyle SP(m)}\downarrow & & \downarrow{\scriptstyle m} \\ SP(A) & \xrightarrow{m} & A \end{array}$$

and

$$\begin{array}{ccc} A & \xrightarrow{\eta^A} & SP(A) \\ & {\scriptstyle Id_A}\searrow & \downarrow{\scriptstyle m} \\ & & A \end{array}$$

commute.