Homotopie et Homologie
Exercise Set 13

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Throughout these exercises, *space* means *topological space* and *map* means *continuous map.*

1. Let $f : (X, x_0) \to (Y, y_0)$ be a map of CW-complexes. Prove that there is a long exact sequence

$$\cdots \xrightarrow{\widetilde{H}_{n+1}i} \widetilde{H}_{n+1}C_f \xrightarrow{\partial_{n+1}} \widetilde{H}_nX \xrightarrow{\widetilde{H}_nf} \widetilde{H}_nY \xrightarrow{\widetilde{H}_ni} \widetilde{H}_nC_f \xrightarrow{\partial_n} \cdots \xrightarrow{\widetilde{H}_1i} \widetilde{H}_1C_f$$

$$\widetilde{H}_1C_f \xrightarrow{\partial_1} \widetilde{H}_0X \xrightarrow{\widetilde{H}_0f} \widetilde{H}_0Y \xrightarrow{\widetilde{H}_0i} \widetilde{H}_0C_f.$$

   This the *long exact sequence in (reduced) homology* associated to a map.

2. Prove that for all CW-pairs (X, A), the sequence

$$\cdots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial_{n+1}} H_nA \longrightarrow H_nX \longrightarrow H_n(X,A) \xrightarrow{\partial_n} \cdots \longrightarrow H_1(X,A)$$

$$H_1(X,A) \xrightarrow{\partial_1} \widetilde{H}_0A \longrightarrow \widetilde{H}_0X \longrightarrow H_0(X,A)$$

   is exact.

   Hint 1. To prove exactness at H_nX, $H_n(X,A)$ and H_nA, apply the sequence of Exercise 1 to the inclusions $A_+ \hookrightarrow X_+$ and $X \hookrightarrow X_+ \coprod_{A_+} CA_+$ and to the quotient map $X_+ \coprod_{A_+} CA_+ \to \Sigma A_+$, respectively.

3. Prove that excision for excisive triads is equivalent to the following property.

   > Let (X, A) be a pair of spaces. For all $U \subseteq A$ satisfying $\overline{U} \subseteq \mathring{A}$, the inclusion $(X \smallsetminus U, A \smallsetminus U) \hookrightarrow (X, A)$ induces an isomorphism
   >
   > $$H_n(X,A) \cong H_n(X \smallsetminus U, A \smallsetminus U) \quad \forall n \geq 0.$$

In other words, one can *excise* a nice enough subset without changing the homology of the pair.

4. Prove the following result, which can be seen as analogous to the Seifert-van Kampen Theorem.

Theorem 2 (Mayer-Vietoris)**.** *If $(X; A, B)$ is an excisive triad, then the sequence*

$$\cdots \xrightarrow{\Delta_{n+1}} H_n(A \cap B) \xrightarrow{\psi_n} H_n(A) \oplus H_n(B) \xrightarrow{\varphi} H_n X \xrightarrow{\Delta_n} \cdots$$

is exact, where

- $\psi_n(c) = \big(H_n(i)(c), H_n(j)(c)\big)$*, where* $i : A \cap B \hookrightarrow A$ *and* $j : A \cap B \hookrightarrow B$ *are the inclusions;*
- $\varphi_n(a, b) = H_n(k)(a) - H_n(\ell)(b)$*, where* $k : A \hookrightarrow X$ *and* $\ell : B \hookrightarrow X$ *are the inclusions; and*
- Δ *is the composite*

$$H_n X \to H_n(X, B) \cong H_n(A, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B),$$

where ∂ *is the connecting homomorphism of the Exactness Axiom.*

5. Recall the definition of complex projective space $\mathbb{C}P^n$ from Exercise 1, Exercise set 12.

   (a) Show that $\mathbb{C}P^n$ admits a CW-decomposition with exactly one $2k$-cell for all $0 \leq k \leq n$ and no odd-dimensional cells.

   (b) Use the Mayer-Vietoris Theorem to compute $H_* \mathbb{C}P^n$.

Joyeuses fêtes de fin d'année!!