Throughout these exercises, space means topological space and map means continuous map.

1. Let \( f : (X, x_0) \to (Y, y_0) \) be a map of CW-complexes. Prove that there is a long exact sequence

\[
\cdots \to \tilde{H}_{n+1} C_f \xrightarrow{\partial_{n+1}} \tilde{H}_n X \xrightarrow{\tilde{H}_n f} \tilde{H}_n Y \xrightarrow{\tilde{H}_n f} \tilde{H}_n C_f \xrightarrow{\partial_n} \cdots \to \tilde{H}_1 C_f \xrightarrow{\partial_1} \tilde{H}_0 X.
\]

This the long exact sequence in (reduced) homology associated to a map.

2. Prove that for all CW-pairs \((X, A)\), the sequence

\[
\cdots \to H_{n+1}(X, A) \xrightarrow{\partial_{n+1}} H_n A \xrightarrow{H_n X} H_n(X, A) \xrightarrow{\partial_n} \cdots \to H_1(X, A) \xrightarrow{\partial_1} H_0(X, A) \xleftarrow{H_0 X} \tilde{H}_0 X
\]

is exact.

Hint 1. To prove exactness at \( H_n X \), \( H_n(X, A) \) and \( H_n A \), apply the sequence of Exercise 1 to the inclusions \( A_+ \hookrightarrow X_+ \) and \( X \hookrightarrow X_+ \coprod_{A_+} C A_+ \) and to the quotient map \( X_+ \coprod_{A_+} C A_+ \to \Sigma A_+ \), respectively.

3. Prove that excision for excisive triads is equivalent to the following property.

Let \((X, A)\) be a pair of spaces. For all \( U \subseteq A \) satisfying \( \overline{U} \subseteq \overline{A} \), the inclusion \((X \setminus U, A \setminus U) \hookrightarrow (X, A)\) induces an isomorphism

\[
H_n(X, A) \cong H_n(X \setminus U, A \setminus U) \quad \forall n \geq 0.
\]
In other words, one can excise a nice enough subset without changing the homology of the pair.

4. Prove the following result, which can be seen as analogous to the Seifert-van Kampen Theorem.

**Theorem 2** (Mayer-Vietoris). If \((X; A, B)\) is an excisive triad, then the sequence

\[
\cdots \xrightarrow{\Delta_{n+1}} \mathbb{H}_n(A \cap B) \xrightarrow{\psi_n} \mathbb{H}_n(A) \oplus \mathbb{H}_n(B) \xrightarrow{\varphi_n} \mathbb{H}_nX \xrightarrow{\Delta_n} \cdots
\]

is exact, where

- \(\psi_n(c) = (\mathbb{H}_n(i)(c), \mathbb{H}_n(j)(c))\), where \(i : A \cap B \hookrightarrow A\) and \(j : A \cap B \hookrightarrow B\) are the inclusions;
- \(\varphi_n(a, b) = \mathbb{H}_n(k)(a) - \mathbb{H}_n(\ell)(b)\), where \(k : A \hookrightarrow X\) and \(\ell : B \hookrightarrow X\) are the inclusions; and
- \(\Delta\) is the composite

\[
\mathbb{H}_nX \xrightarrow{\partial} \mathbb{H}_n(X, B) \cong \mathbb{H}_n(A, A \cap B) \xrightarrow{\partial} \mathbb{H}_{n-1}(A \cap B),
\]

where \(\partial\) is the connecting homomorphism of the Exactness Axiom.

5. Recall the definition of complex projective space \(\mathbb{C}P^n\) from Exercise 1, Exercise set 12.

(a) Show that \(\mathbb{C}P^n\) admits a CW-decomposition with exactly one \(2k\)-cell for all \(0 \leq k \leq n\) and no odd-dimensional cells.

(b) Use the Mayer-Vietoris Theorem to compute \(H_*\mathbb{C}P^n\).

Joyeuses fêtes de fin d’année!!