Homotopie et Homologie
Exercise Set 2

29.09.2011

Throughout these exercises, I denotes the unit interval $[0, 1]$, $\cong$ denotes homeomorphism of topological spaces, and *space* means *topological space*.

1. Let X be a Hausdorff space, Y a locally compact Hausdorff space and Z any space. Show that

$$\big\{\mathscr{O}_{K,\mathscr{O}_{L,U}} \mid K \subseteq X \text{ compact }, L \subseteq Y \text{ compact }, U \subseteq Z \text{ open }\big\}$$

is a sub-basis for the compact-open topology on $\mathrm{Map}\big(X, \mathrm{Map}(Y, Z)\big)$.

2. Let X and Y be locally compact, Hausdorff spaces, and let Z be any topological space. Show that composition of functions restricts to a continuous map

$$\gamma : \mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z) : (f, g) \mapsto g \circ f.$$

Explain how the continuity of γ implies that any continuous map $f : X \to Y$ induces a continuous map

$$f^\sharp : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z) : g \mapsto g \circ f$$

and any continuous map $g : Y \to Z$ induces a continuous map

$$g_\sharp : \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z).$$

3. Let X and Y be spaces, and let $A \subseteq X$ and $B \subseteq Y$ be subspaces. Set

$$\mathrm{Map}\big((X, A), (Y, B)\big) := \big\{f \in \mathrm{Map}(X, Y) \mid f(A) \subseteq B\big\},$$

endowed with the subspace topology, and set

$$(X, A) \times (Y, B) := \big(X \times Y, (X \times B) \cup (A \times Y)\big).$$

Establish a *relative version* of the "exponential law" proved in class, i.e., for all $A \subseteq X$, $B \subseteq Y$ and $C \subseteq Z$

(a) the function $\alpha : \mathrm{Map}(X \times Y, Z) \to \mathrm{Map}\big(X, \mathrm{Map}(Y, Z)\big)$ restricts to a function

$$\alpha : \mathrm{Map}\big((X, A)\times(Y, B), (Z, C)\big) \to \mathrm{Map}\Big((X, A), \mathrm{Map}\big((Y, B), (Z, C)\big)\Big),$$

which is continous if X is Hausdorff and Y is locally compact and Hausdorff; and

(b) if Y is locally compact and Hausdorff, the function $\beta : \mathrm{Map}\big(X, \mathrm{Map}(Y, Z)\big) \to \mathrm{Map}(X \times Y, Z)$ restricts to a function

$$\beta : \mathrm{Map}\Big((X, A), \mathrm{Map}\big((Y, B), (Z, C)\big)\Big) \to \mathrm{Map}\big((X, A)\times(Y, B), (Z, C)\big),$$

which is continuous if X is also Hausdorff.

Conclude that

$$\mathrm{Map}\big((X, A) \times (Y, B), (Z, C)\big) \cong \mathrm{Map}\Big((X, A), \mathrm{Map}\big((Y, B), (Z, C)\big)\Big)$$

as long as X is Hausdorff, and Y is locally compact and Hausdorff.

4. Let X be a topological space, and let $x_0 \in X$.

(a) Prove by induction that $\big(I^{\times n}, \partial(I^{\times n})\big) = (I, \partial I)^{\times n}$ for all $n \in \mathbb{N}$.

(b) Prove that $\partial(I^{\times n+1})$ is homemorphic to the n-sphere

$$S^n = \{z \in \mathbb{R}^{n+1} \mid ||z|| = 1\}.$$

(c) The n^{th}*-loop space of* X *based at* x_0 is

$$\Omega^n(X, x_0) := \mathrm{Map}\big((I^{\times n}, \partial(I^{\times n})), (X, x_0)\big).$$

Show that

$$\Omega^n(X, x_0) \cong \Omega\big(\Omega^{n-1}(X, x_0), c_{x_0}\big),$$

where $c_{x_0} : I^{\times n} \to X$ sends every element of $I^{\times n}$ to x_0, and that

$$\Omega^n(X, x_0) \cong \mathrm{Map}\big((S^n, z_0), (X, x_0)\big),$$

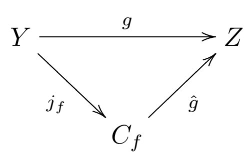
where z_0 is any point of S^n.

5. Let $f : (X, x_0) \to (Y, y_0)$ be a pointed continuous map. The *mapping cone* of f is the space

$$C_f = Y \cup_f CX,$$

where CX is the reduced cone on X. Let $j_f : Y \to C_f$ denote the canonical inclusion.

Let $g : (Y, y_0) \to (Z, z_0)$ be another pointed continuous map. Show that $g \circ f$ is nullhomotopic if and only if there exists $\hat{g} : C_f \to Z$ such that



commutes.