Homotopie et Homologie
Exercise Set 7

10.11.2011

Throughout these exercises, *space* means *topological space*, *map* means *continuous map*, and I denotes $[0,1]$.
The goal of this series of exercises is to develop and study the dual Barrat-Puppe sequence. We begin by defining the constructions that are analogous to the cone and mapping cone constructions.

Definition 1. The *based path space* on a pointed space (Y, y_0), denoted PY, is the *pullback* of

$$\{y_0\} \hookrightarrow Y \xleftarrow{ev_0} \mathrm{Map}(I, Y),$$

where $ev_0(\lambda) = \lambda(0)$, i.e., $PY = \big\{\lambda \in \mathrm{Map}(I, Y) \mid \lambda(0) = y_0\big\}$. Let $e : PY \to Y$ denote the map given by $e(\lambda) = \lambda(1)$.

Definition 2. The *homotopy fiber* of a map $f : X \to Y$, denoted P_f, is the *pullback* of

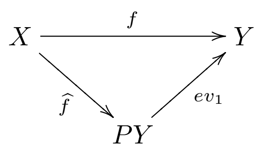
$$X \xrightarrow{f} Y \xleftarrow{e} PY,$$

i.e., $P_f = \big\{(x, \lambda) \in X \times PY \mid \lambda(1) = x\big\}$, which is a subspace of $X \times PY$. Let $q_f : P_f \to X : (x, \lambda) \to x$.

1. Prove that a pointed map $f : (X, x_0) \to (Y, y_0)$ is nullhomotopic if and only if there is a pointed map

   $$\widehat{f} : (X, x_0) \to (PY, c_{y_0})$$

   such that

   commutes.

2. Let $(W, w_0) \xrightarrow{g} (X, x_0) \xrightarrow{f} (Y, y_0)$ be pointed maps. Prove that $f \circ g$ is nullhomotopic if and only if there exists a pointed map

$$\widehat{g} : (W, w_0) \to \big(P_f, (x_0, c_{y_0})\big)$$

such that

$$\begin{array}{ccc} W & \xrightarrow{g} & X \\ & {\scriptstyle \widehat{g}} \searrow \quad \nearrow {\scriptstyle q_f} & \\ & P_f & \end{array}$$

commutes.

3. Identify the homotopy fibers of the inclusion of the basepoint $\{y_0\}$ into Y, of $e : PY \to Y$ and of $q_f : P_f \to X$, for any pointed map $f : (X, x_0) \to (Y, y_0)$. Show that $\Omega P_f \simeq_* P_{\Omega f}$ as well.

4. Prove that e satisfies the *homotopy lifting property*, i.e., for every commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & PY \\ {\scriptstyle j_0} \downarrow & & \downarrow {\scriptstyle e} \\ X \times I & \xrightarrow{F} & Y \end{array} ,$$

where $j_0(x) = (x, 0)$, there is a map $\widehat{F} : X \times I \to PY$ such that $e \circ \widehat{F} = F$ and $\widehat{F} \circ j_0 = f$.

5. Prove that for any pointed map $f : (X, x_0) \to (Y, y_0)$ (respectively, H-morphism $f : (X, x_0, \mu) \to (Y, y_0, \nu)$), the sequence $P_f \xrightarrow{q_f} X \xrightarrow{f} Y$ is h-exact, i.e., for any pointed space (W, w_0), the sequence of set maps (respectively, of homomorphisms)

$$[W, P_f]_* \xrightarrow{(q_f)_*} [W, X]_* \xrightarrow{f_*} [W, Y]_*$$

is exact.

6. Given a pointed map $f : (X, x_0) \to (Y, y_0)$, explain how to obtain a sequence of pointed maps

$$\cdots \to \Omega^2 X \to \Omega^2 Y \to P_{\Omega f} \to \Omega X \to \Omega Y \to P_f \to X \to Y$$

(basepoints suppressed) by iterating the homotopy fiber construction: the *dual Barratt-Puppe sequence*. Conclude, using exercise 5, that for any pointed space (W, w_0), the sequence

$$\cdots \to [W, P_{\Omega f}]_* \to [W, \Omega X] \to [W, \Omega Y]_* \to [W, P_f]_* \to [W, X]_* \to [W, Y]_*$$

is exact.