Homotopie et Homologie
Exercise Set 8

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Throughout these exercises, *space* means *topological space*, *map* means *continuous map*, and I denotes $[0,1]$.

1. Let A be a subspace of X, and let $x_0 \in A$. Let P_i denote the homotopy fiber of the inclusion $i : A \hookrightarrow X$ (cf. Exercise Set 6). Show that
$$\pi_n(X, A) \cong \pi_{n-1} P_i.$$

2. Let $f : X \to Y$ be any pointed map, and let P_f denote the homotopy fiber of f. Prove that there is an exact sequence
$$\cdots \to \pi_n P_f \to \pi_n X \to \pi_n Y \to \pi_{n-1} P_f \to \cdots \to \pi_1 Y \to \pi_0 P_f \to \pi_0 X \to \pi_0 Y$$
and describe the homomorphisms in the sequence. This is the *long exact sequence in homotopy* of the *fiber sequence* $P_f \xrightarrow{q_f} X \xrightarrow{f} Y$.

   Hint 1. *The existence of this long exact sequence can be proved using either the dual Barratt-Puppe sequence or the long exact sequence of a pair. Try to do it both ways!*

3. Prove that if X is contractible, then $\pi_{n-1} P_f \cong \pi_n Y$ for all pointed maps $f : X \to Y$.

4. Let $\{x_0\} \subseteq A \subseteq B \subseteq X$ be a sequence of subspace inclusions. Show that there is an exact sequence
$$\cdots \to \pi_n(B, A) \to \pi_n(X, A) \to \pi_n(X, B) \xrightarrow{\partial_n} \pi_{n-1}(B, A) \to \cdots .$$

   Hint 2. *The* connecting homomorphism ∂_n *is equal to the composite*
$$\pi_n(X, B) \to \pi_{n-1} B \to \pi_{n-1}(B, A).$$

5. Consider the map $h : \mathbb{R}^4 \to \mathbb{R}^3$ given by
$$h(x_1, x_2, x_3, x_4) = (x_1 x_3 + x_2 x_4, x_2 x_3 - x_1 x_4, x_1^2 + x_2^2 - x_3^2 - x_4^2).$$

(a) Show that h restricts and corestricts to a map $h : S^3 \to S^2$. (This is the famous *Hopf map.*)

(b) Show that the homotopy fiber P_h of h has the homotopy type of S^1.

Hint 3. *Start by showing that* $h^{-1}(v) \cong S^1$ *for any* $v \in S^2$.

(c) Show that $\pi_n S^1 = \{0\}$ for all $n > 1$.

Hint 4. *Use the relation* $\pi_n X \cong \pi_{n-1} \Omega X$.

(d) It follows from the *Freudenthal suspension theorem* (to be proved later this semester) that

$$\pi_k S^n \cong \pi_{k+1} S^{n+1} \quad \forall k < 2n - 1.$$

Use this isomorphism to prove that $\pi_k S^n = \{0\}$ for all $1 \leq k < n$.

Hint 5. *Recall that Seifert-van Kampen implies that* S^n *is simply connected for all* $n \geq 2$.

(e) Show that $\pi_2 S^2 \cong \mathbb{Z}$ (whence, by Freudenthal, $\pi_n S^n \cong \mathbb{Z}$ for all n) and that $\pi_3 S^2 \cong \mathbb{Z}$.

Hint 6. *Use the long exact sequence in homotopy of* h.