Homotopie et Homologie
Exercise Set 9

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Throughout these exercises, *space* means *topological space*, *map* means *continuous map*, and I denotes $[0,1]$.

1. Let $j : A \hookrightarrow X$ denote the inclusion of a subspace.
   (a) Prove that if j is a cofibration, then so is
   $$j \times Id_Z : A \times Z \to X \times Z$$
   for all spaces Z.
   (b) Conclude that for all maps $f : A \to Y$, the induced inclusion
   $$Y \hookrightarrow Y \cup_f X = Y \coprod X/\sim,$$
   where $a \sim f(a)$ for all $a \in A$, is also a cofibration.

2. Prove that if $i : A \hookrightarrow B$ and $j : B \hookrightarrow X$ are cofibrations, then so is $j \circ i : A \hookrightarrow X$. Prove more generally that if
   $$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} \cdots$$
   are inclusions that are cofibrations, then the inclusion $A_0 \hookrightarrow \bigcup_{n\geq 0} A_n$ is a cofibration, if the topology on $\bigcup_{n\geq 0} A_n$ satisfies:
   $$C \subseteq \bigcup_{n\geq 0} A_n \text{ closed } \Leftrightarrow C \cap A_n \text{ closed in } A_n \forall n.$$

3. (An interesting class of cofibrations.) Let X be a space, with subspace $A \subset V$. If there is a homotopy $H : V \times I \to X$ such that
   - $H(v,0) = v$ for all $v \in V$,
   - $H(a,t) = a$ for all $a \in A$ and $t \in I$, and
   - $H(v,1) \in A$ for all $v \in V$,

then A is a *strong deformation retract (SDR)* of V in X.

A space X is *perfectly normal* if for every pair of disjoint closed sets C and D in X, there is a continous map $\varphi : X \to I$ such that $C = \varphi^{-1}(0)$ and $D = \varphi^{-1}(1)$.

(a) Prove that S^n is an SDR of $D^{n+1} \smallsetminus \{0\}$ in D^{n+1} for all $n \geq 0$.

(b) Prove that every metrizable space is perfectly normal.

(c) Let X be a perfectly normal space, and let A be a subspace of X. Prove that the inclusion $A \hookrightarrow X$ is a cofibration if there is an open subset V of X such that A is an SDR of V in X.

4. (Important examples of cofibrations.)

   (a) Prove that for all maps $f : S^n \to Y$, the inclusion $Y \hookrightarrow Y \cup_f D^{n+1}$ is a cofibration. (Thus, for example, $S^1 \hookrightarrow \mathbb{R}P^2$ is a cofibration.)

   (b) Prove that for all maps $f : X \to Y$, the inclusion $i : Y \hookrightarrow C_f$ is a cofibration.

5. Prove that an inclusion $j : A \hookrightarrow X$ is a cofibration if and only if the map

$$\mathrm{Map}(X \times I, Y) \to \mathrm{Map}(X, Y) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(A \times I, Y)$$

induced by

$$j_0^* : \mathrm{Map}(X \times I, Y) \to \mathrm{Map}(X, Y) : H \mapsto H \circ j_0$$

and

$$(j \times Id_I)^* : \mathrm{Map}(X \times I, Y) \to \mathrm{Map}(A \times I, Y) : H \mapsto H \circ (j \times Id_I)$$

is surjective.