Homotopie et Homologie
Exercise Set 1

23.09.2010

Throughout these exercises, I denotes the unit interval $[0,1]$, $\cong$ denotes homeomorphism of topological spaces, and *space* means *topological space*.

1. Let X be a Hausdorff space, Y a locally compact Hausdorff space and Z any space. Show that

$$\big\{\mathscr{O}_{K,\mathscr{O}_{L,U}} \mid K \subseteq X \text{ compact }, L \subseteq Y \text{ compact }, U \subseteq Z \text{ open } \big\}$$

is a sub-basis for the compact-open topology on $\mathrm{Map}\big(X, \mathrm{Map}(Y,Z)\big)$.

2. Let X and Y be locally compact, Hausdorff spaces, and let Z be any topological space. Show that composition of functions restricts to a continuous map

$$\gamma : \mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z) \to \mathrm{Map}(X,Z) : (f,g) \mapsto g \circ f.$$

Explain how the continuity of γ implies that any continuous map $f : X \to Y$ induces a continuous map

$$f^\sharp : \mathrm{Map}(Y,Z) \to \mathrm{Map}(X,Z) : g \mapsto g \circ f$$

and any continuous map $g : Y \to Z$ induces a continuous map

$$g_\sharp : \mathrm{Map}(X,Y) \to \mathrm{Map}(X,Z).$$

3. Let X and Y be spaces, and let $A \subseteq X$ and $B \subseteq Y$ be subspaces. Set

$$\mathrm{Map}\big((X,A),(Y,B)\big) := \big\{f \in \mathrm{Map}(X,Y) \mid f(A) \subseteq B\big\},$$

endowed with the subspace topology, and set

$$(X,A) \times (Y,B) := \big(X \times Y, (X \times B) \cup (A \times Y)\big).$$

Establish a *relative version* of the "exponential law" proved in class, i.e., for all $A \subseteq X$, $B \subseteq Y$ and $C \subseteq Z$

(a) the function $\alpha : \mathrm{Map}(X \times Y, Z) \to \mathrm{Map}\big(X, \mathrm{Map}(Y, Z)\big)$ restricts to a function

$$\alpha : \mathrm{Map}\big((X,A)\times(Y,B),(Z,C)\big) \to \mathrm{Map}\Big((X,A), \mathrm{Map}\big((Y,B),(Z,C)\big)\Big),$$

which is continous if X is Hausdorff and Y is locally compact and Hausdorff; and

(b) if Y is locally compact and Hausdorff, the function $\beta : \mathrm{Map}\big(X, \mathrm{Map}(Y, Z)\big) \to \mathrm{Map}(X \times Y, Z)$ restricts to a function

$$\beta : \mathrm{Map}\Big((X,A), \mathrm{Map}\big((Y,B),(Z,C)\big)\Big) \to \mathrm{Map}\big((X,A)\times(Y,B),(Z,C)\big),$$

which is continuous if X is also Hausdorff.

Conclude that

$$\mathrm{Map}\big((X,A)\times(Y,B),(Z,C)\big) \cong \mathrm{Map}\Big((X,A), \mathrm{Map}\big((Y,B),(Z,C)\big)\Big)$$

as long as X is Hausdorff, and Y is locally compact and Hausdorff.

4. Let X be a topological space, and let $x_0 \in X$.

(a) Prove by induction that $\big(I^{\times n}, \partial(I^{\times n})\big) = (I, \partial I)^{\times n}$ for all $n \in \mathbb{N}$.

(b) Prove that $\partial(I^{\times n+1})$ is homemorphic to the n-sphere

$$S^n = \{z \in \mathbb{R}^{n+1} \mid ||z|| = 1\}.$$

(c) The n^{th}*-loop space of* X *based at* x_0 is

$$\Omega^n(X, x_0) := \mathrm{Map}\big((I^{\times n}, \partial(I^{\times n})), (X, x_0)\big).$$

Show that

$$\Omega^n(X, x_0) \cong \Omega\big(\Omega^{n-1}(X, x_0), c_{x_0}\big),$$

where $c_{x_0} : I^{\times n} \to X$ sends every element of $I^{\times n}$ to x_0, and that

$$\Omega^n(X, x_0) \cong \mathrm{Map}\big((S^n, z_0), (X, x_0)\big),$$

where z_0 is any point of S^n.

5. Let X be any space.

(a) Consider the relation

$$R_{\text{path}} = \{(x, x') \in X \times X \mid \exists \lambda \in \mathrm{Map}(I, X) \text{ s.t. } \lambda(0) = x, \lambda(1) = x'\}$$

on X. Show that R_{path} is an equivalence relation.
Henceforth, we write $x \sim x'$ if $(x, x') \in R$, and denote the equivalence class of x by $[x]$.

(b) The *set of path-components of* X is the set

$$\pi_0 X := X/\sim$$

of equivalence classes of elements in X under the relation R_{path}. Show that if $f : X \to Y$ is continuous map, then

$$\pi_0 f : \pi_0 X \to \pi_0 Y : [x] \mapsto \big[f(x)\big]$$

is a well-defined function.

(c) Prove that if $f : X \to Y$ is a homeomorphism, then $\pi_0 f$ is a bijection.

(d) Prove that for all spaces X and Y, there is a bijection

$$\tau_{X,Y} : \pi_0 X \times \pi_0 Y \to \pi_0(X \times Y)$$

such that

$$\begin{array}{ccc} \pi_0 X \times \pi_0 Y & \xrightarrow{\tau_{X,Y}} & \pi_0(X \times Y) \\ {\scriptstyle \pi_0 f \times \pi_0 g}\downarrow & & \downarrow{\scriptstyle \pi_0(f\times g)} \\ \pi_0 X' \times \pi_0 Y' & \xrightarrow{\tau_{X',Y'}} & \pi_0(X' \times Y') \end{array}$$

commutes for all continuous maps $f : X \to X'$ and $g : Y \to Y'$.

6. Let X be a locally compact, Hausdorff space, and let Y be any topological space. Let $A \subseteq X$ and $B \subseteq Y$.

(a) Show that

$$\big[(X, A), (Y, B)\big] = \pi_0 \operatorname{Map}\big((X, A), (Y, B)\big).$$

(b) Show that for any continuous map $f : X \to X'$, where X' is locally compact, Hausdorff and $f(A) \subseteq A' \subseteq X'$,

$$\pi_0 f^\sharp = f^* : \big[(X', A'), (Y, B)\big] \to \big[(X, A), (Y, B)\big];$$

(c) Show that if Y is locally compact, Hausdorff, then for any continuous map $g : Y \to Y'$ such that $g(B) \subseteq B' \subseteq Y'$,

$$\pi_0 g_\sharp = g_* : \big[(X, A), (Y, B)\big] \to \big[(X, A), (Y', B')\big];$$

(d) More generally, show that the composite

$$\pi_0 \operatorname{Map}(X, Y) \times \pi_0 \operatorname{Map}(Y, Z) \xrightarrow{\tau} \pi_0\big(\operatorname{Map}(X, Y) \times \operatorname{Map}(Y, Z)\big) \xrightarrow{\pi_0 \gamma} \pi_0 \operatorname{Map}(X, Z)$$

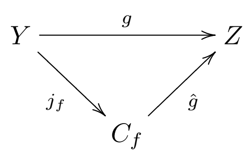
agrees with the definition of composition of homotopy classes of maps.

7. Let $\{X_j \mid j \in \mathscr{J}\}$ be a collection of spaces, and let $A_j \subseteq X_j$ for all $j \in \mathscr{J}$. Let $(X, A) = \big(\coprod_{j\in\mathscr{J}} X_j, \coprod_{j\in\mathscr{J}} A_j\big)$, where $\coprod$ denotes disjoint union.

   Show that there is a bijection
$$\big[(X, A), (Y, B)\big] \to \prod_{j\in J} \big[(X_j, A_j), (Y, B)\big]$$
   for all spaces Y and subspaces $B \subseteq Y$.

8. Let $f : (X, x_0) \to (Y, y_0)$ be a pointed continuous map. The *mapping cone* of f is the space
$$C_f = Y \cup_f CX,$$
   where CX is the reduced cone on X. Let $j_f : Y \to C_f$ denote the canonical inclusion.

   Let $g : (Y, y_0) \to (Z, z_0)$ be another pointed continuous map. Show that $g \circ f$ is nullhomotopic if and only if there exists $\hat{g} : C_f \to Z$ such that



   commutes.