Homotopie et Homologie
Exercise Set 10

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Throughout these exercises, *space* means *topological space*, *map* means *continuous map*, and I denotes $[0,1]$.

1. Show that if X and Y are CW-complexes, and X and Y both have countably many cells, then $X \times Y$ is also a CW-complex.

   Hint 1. For all $0 \leq k \leq n$,

   $$(D^n, S^{n-1}) \cong (D^k \times D^{n-k}, S^{k-1} \times D^{n-k} \cup D^k \times S^{n-k-1}).$$

2. Let A be a subcomplex of a CW-complex X.

   (a) Show that the inclusion map $A \to X$ is a cofibration.

   (b) Show that X/A is also a CW-complex.

3. The *smash product* of two pointed spaces (X, x_0) and (Y, y_0) is the space

   $$X \wedge Y = X \times Y / X \vee Y,$$

   with basepoint equal to the equivalence class of (x_0, y_0).

   (a) Show that $S^1 \wedge X \cong \Sigma X$, for all (X, x_0).

   (b) Let X and Y be CW-complexes such that $X_{r-1} = \{x_0\}$ and $Y_{s-1} = \{y_0\}$. Show that $X \wedge Y$ is a CW-complex with $(X \wedge Y)_{r+s-1} = \{[x_0, y_0]\}$.

   (c) Conclude that $\Sigma^n X$ is at least $(n-1)$-connected, for all (X, x_0).

4. Prove that a map $f : X \to Y$ is an n-equivalence if and only if the pair (M_f, X) is n-connected, where M_f is the mapping cylinder of f, i.e., the pushout of $X \times I \xleftarrow{j_0} X \xrightarrow{f} Y$, and X is seen as a subspace of M_f by inclusion into the top of the cylinder.

5. Prove that the mapping cylinder of a cellular map is a CW-complex.

6. Let G be a discrete group acting on the right on a space X, i.e., there is a map $\rho : X \times G \to X : (x, a) \mapsto x \cdot a$ such that $(x \cdot a) \cdot b = x \cdot (ab)$ for all $x \in X$ and $a, b \in G$ and $x \cdot e = x$ for all $x \in X$, where e is the neutral element of G. Show that if X is a CW-complex, and ρ is cellular, then the orbit space X/G is a CW-complex.

7. Let $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\} \subset \mathbb{R}$. Show that X is not homotopy equivalent to a CW-complex.