Throughout these exercises, space means topological space, map means continuous map and $I$ denotes $[0,1]$.

1. For any finite CW-complex $X$, let $\zeta_n(X)$ denote the number of $n$-cells of $X$. The Euler characteristic of $X$ is then

$$\chi(X) := \sum_{n \geq 0} (-1)^n \zeta_n(X).$$

Let $A$ be a subcomplex of $X$, and let $f : A \to Y$ be a cellular map, where $Y$ is also finite.

(a) Show that the pushout $X \amalg_A Y$ of $Y \xleftarrow{f} A \xrightarrow{i} X$ is a CW-complex. (Note: the finiteness assumptions are not necessary here.)

(b) Find and prove a formula for $\chi(X \amalg_A Y)$ in terms of $\chi(A)$, $\chi(X)$ and $\chi(Y)$.

2. The Lusternik-Schnirelmann category of a path-connected, pointed space $(X,x_0)$, denoted $\text{cat}(X)$, is the least integer $n$ such that $X$ admits an open cover $\{U_0,...,U_n\}$ where each $U_i$ is contractible in $X$, i.e., there exist a homotopies $H_i : X \times I \to X$ such that $H_i(-,0) = 1d_X$ and $H_i(u,1) = x_0$ for all $u \in U_i$.

(a) Prove that Lusternik-Schnirelmann category is a homotopy invariant.

(b) Prove that if $X$ is a CW-complex of dimension $n$, then $\text{cat}(X) \leq n$.

3. We see in this exercise how to construct a canonical CW-approximation to a space $X$.

For any $n \in \mathbb{N}$, let

$$\Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \forall i\},$$

and let

$$\partial^i : \Delta^{n-1} \to \Delta^n : (t_0, ..., t_{n-1}) \mapsto (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1})$$
and
\[ \sigma^i : \Delta^{n+1} \rightarrow \Delta^n : (t_0, \ldots, t_{n+1}) \mapsto (t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1}) \]
for all \( 0 \leq i \leq n \).

For any space \( X \), let
\[ S_n(X) = \{ \varphi : \Delta^n \rightarrow X \mid \sigma \text{ continuous} \}, \]
seen as a discrete topological space, and
\[ \Gamma X = \coprod_{n \geq 0} S_n(X) \times \Delta^n / \sim, \]
where
\[ (\varphi \circ \sigma^i, t) \sim (\varphi, \sigma^i(t)) \quad \text{and} \quad (\varphi \circ \partial^i, s) \sim (\varphi, \partial^i(s)) \]
for all \( \varphi \in S_n(X) \), \( t \in \Delta^{n+1} \), \( s \in \Delta^{n-1} \), \( 0 \leq i \leq n \) and \( n \geq 0 \). Endow \( \Gamma X \)
with the obvious quotient topology.

(a) Show that there is a CW-structure on \( \Gamma X \) such that the quotient
map \( \coprod_{n \geq 0} S_n(X) \times \Delta^n \rightarrow \Gamma X \) is a cellular map.

(b) Show that the evaluation maps \( \text{ev} : S_n(X) \times \Delta^n \rightarrow X : (\varphi, t) \mapsto \varphi(t) \)
together give rise to a continuous map \( \varepsilon_X : \Gamma X \rightarrow X \).

(c) Show that any map \( f : X \rightarrow Y \) gives rise to a map \( \Gamma f : \Gamma X \rightarrow \Gamma Y \)
such that
\[
\begin{array}{ccc}
\Gamma X & \xrightarrow{\Gamma f} & \Gamma Y \\
\varepsilon_X & \downarrow & \varepsilon_Y \\
X & \xrightarrow{f} & Y
\end{array}
\]
commutes.

Remark 1. The map \( \varepsilon_X \) behaves very nicely: it satisfies a universal property (which we do not spell out here), and it is always a weak equivalence!