Throughout these exercises, $I$ denotes the unit interval $[0, 1]$, $\cong$ denotes homeomorphism of topological spaces or isomorphism of groups, and space means topological space.

1. For any positive integer $n$, explain how to construct a path-connected space $X_n$ from a circle and a disk such that $\pi_1(X_n, x_0) \cong \mathbb{Z}/n\mathbb{Z}$. Show that $X_2$ is homeomorphic to $\mathbb{R}P^2$, the real projective plane, which is usually defined as the quotient of $D^2$ by the relation that identifies antipodal points on its boundary.

2. Prove that $G \ast \{e\} \cong G$ for every group $G$.

3. Let $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$ be group homomorphisms. The pushout (or amalgamated sum) of $f_1$ and $f_2$ consists of a group $H$ and two homomorphisms $g_1 : G_1 \to H$ and $g_2 : G_2 \to H$ such that $g_1 f_1 = g_2 f_2$ and such that for any pair of homomorphisms $h_1 : G_1 \to K$ and $h_2 : G_2 \to K$ such that

\[
\begin{array}{ccc}
G_0 & \xrightarrow{g_1} & G_1 \\
| & g_2 | & | \\
G_2 & \xrightarrow{h_2} & K \\
\end{array}
\]

commutes, there is a unique homomorphism $h : H \to K$ such that

\[
\begin{array}{ccc}
G_1 & \xrightarrow{g_1} & H \\
| & h_1 | & | \\
K & \xleftarrow{h_2} & G_2 \\
\end{array}
\]

commutes. Prove that pushouts of pairs of group homomorphisms exist and are unique up to isomorphism.

4. Let $h : W \to X$ and $k : W \to Y$ be continuous maps. The pushout of $h$ and $k$ is the quotient space

\[
X \bigsqcup_W Y = X \bigsqcup Y / \sim,
\]
where \( \sim \) is the equivalence relation generated by \( h(w) \sim k(w) \) for all \( w \in W \). Show that for any pair of continuous maps \( f : X \to Z \) and \( g : Y \to Z \) such that

\[
\begin{array}{ccc}
W & \xrightarrow{h} & X \\
\downarrow{k} & & \downarrow{f} \\
Y & \xrightarrow{g} & Z
\end{array}
\]

commutes, there is a unique continuous map \( f + g : X \sqcup Y \to Z \) such that

\[
\begin{array}{ccc}
X & \xrightarrow{i_X} & X \sqcup Y \\
\downarrow{f} & & \downarrow{f + g} \\
Y & \xleftarrow{i_Y} & Z \\
\end{array}
\]

commutes. Formulate and prove a pointed version of this result.

5. Let \( f : X \to Z \) and \( g : Y \to Z \) be continuous maps. The pullback of \( f \) and \( g \) is the space

\[ X \times_Y \{ (x, y) \in X \times Y \mid f(x) = g(y) \}, \]

topologized as a subspace of \( X \times Y \). Show that for any pair of continuous maps \( h : W \to X \) and \( k : W \to Y \) such that

\[
\begin{array}{ccc}
W & \xrightarrow{h} & X \\
\downarrow{k} & & \downarrow{f} \\
Y & \xrightarrow{g} & Z
\end{array}
\]

commutes, there is a unique continuous map \( (h, k) : W \to X \times_Y \{ (x, y) \in X \times Y \mid f(x) = g(y) \} \) such that

\[
\begin{array}{ccc}
W & \xrightarrow{h} & X \times_Y \{ (x, y) \in X \times Y \mid f(x) = g(y) \} \\
\downarrow{(h, k)} & & \downarrow{\text{proj}_Y} \\
X & \xleftarrow{\text{proj}_X} & Y \\
\end{array}
\]

commutes. Formulate and prove a pointed version of this result.