Throughout these exercises, space means topological space.

1. Let \((X,x_0,\psi)\) be a co-\(H\)-space. Show that if \(f : (X,x_0) \to (X',x'_0)\) is a based homotopy equivalence, then \(X'\) admits a co-\(H\)-space structure with respect to which \(f\) is a co-\(H\)-morphism. Show that, moreover, if \(X\) is actually a co-\(H\)-group, then so is \(X'\). In other words, “to be a co-\(H\)-space (respectively, co-\(H\)-group)” is a homotopy invariant notion.

2. Let \((X,x_0,\psi)\) be a co-\(H\)-space, and let \((Y,y_0)\) be any based space. Show that \(\text{Map}((X,x_0),(Y,y_0))\) admits an \(H\)-space structure, which is natural in the sense that if \(f : (Y,y_0) \to (Z,z_0)\) is a based, continuous map, then the induced map

\[ f_* : \text{Map}((X,x_0),(Y,y_0)) \to \text{Map}((X,x_0),(Z,z_0)) \]

is an \(H\)-morphism.

3. Let \((X,x_0)\) be any based space. Check that the comultiplication and coinverse maps on \(\Sigma(X,x_0)\) defined in class do indeed satisfy the axioms of a co-\(H\)-group.

4. It is easy to see that there are \(H\)-groups with strictly associative multiplication, strict units and strict inverses: any topological group \((S^1, GL(n,\mathbb{R}), O(n), U(n),...)\) is an example of such. Show that, on the other hand, there are no nontrivial, strict co-\(H\)-spaces.

5. For any based, continuous map \(f : (X,x_0) \to (Y,y_0)\), show that \(C_f/Y \cong \Sigma(X,x_0)\) (cf. Exercise Set 1, # 8).

6. Let \(f : (X,x_0) \to (X',x'_0)\) be a pointed, continuous map. Explain how to define \(\Sigma f : \Sigma(X,x_0) \to \Sigma(X',x'_0)\) and \(\Omega f : \Omega(X,x_0) \to \Omega(X',x'_0)\) so that \(\Sigma f\) is a co-\(H\)-morphism and \(\Omega f\) is an \(H\)-morphism.

7. For “any” pointed space \((X,x_0)\), define pointed, continuous maps

\[ \eta_{(X,x_0)} : (X,x_0) \to \Omega \Sigma(X,x_0) \]
and

\[ \epsilon(X, x_0) : \Sigma \Omega(X, x_0) \to (X, x_0) \]
such that for all pointed, continuous maps \( f : (X, x_0) \to (X', x'_0) \),

\[ \Omega \Sigma f \circ \eta(X, x_0) = \eta(X', x'_0) \circ f : (X, x_0) \to \Omega \Sigma(X', x'_0), \]

and

\[ f \circ \epsilon(X, x_0) = \epsilon(X', x'_0) \circ \Sigma f : \Sigma \Omega(X, x_0) \to (X', x'_0), \]

while

\[ \alpha : \text{Map}(\Sigma(X, x_0), (X', x'_0)) \to \text{Map}((X, x_0), \Omega(X', x'_0)) : g \mapsto \Omega g \circ \eta(X, x_0) \]

and

\[ \beta : \text{Map}((X, x_0), \Omega(X', x'_0)) \to \text{Map}(\Sigma(X, x_0), (X', x'_0)) : g \mapsto \epsilon(X', x'_0) \circ \Sigma g \]

are mutually inverse homeomorphisms.