

## ***G*-CHAPLYGIN SYSTEMS WITH INTERNAL SYMMETRIES, TRUNCATION, AND AN (ALMOST) SYMPLECTIC VIEW OF CHAPLYGIN'S BALL**

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ABSTRACT. Via compression ([18, 8]) we write the  $n$ -dimensional Chaplygin sphere system as an almost Hamiltonian system on  $T^*\mathrm{SO}(n)$  with internal symmetry group  $\mathrm{SO}(n-1)$ . We show how this symmetry group can be factored out, and pass to the fully reduced system on (a fiber bundle over)  $T^*S^{n-1}$ . This approach yields an explicit description of the reduced system in terms of the geometric data involved. Due to this description we can study Hamiltonizability of the system. It turns out that the homogeneous Chaplygin ball, which is not Hamiltonian at the  $T^*\mathrm{SO}(n)$ -level, is Hamiltonian at the  $T^*S^{n-1}$ -level. Moreover, the 3-dimensional ball becomes Hamiltonian at the  $T^*S^2$ -level after time reparametrization, whereby we re-prove a result of [4, 5] in symplectogeometric terms. We also study compression followed by reduction of generalized Chaplygin systems.

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**1. Introduction and description of results.** A non-holonomic system (with linear constraints) consists of a configuration manifold  $Q$ , a Lagrangian  $L : TQ \rightarrow \mathbb{R}$ ,

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and a non-integrable smooth distribution  $\mathcal{D} \subset TQ$ . The equations of motion for a curve  $q(t)$  in  $Q$  are determined by the Lagrange-d'Alembert principle (constraining force does not exert work) supplemented by the condition that  $q' \in \mathcal{D}$ . We shall only deal with constraint distributions  $\mathcal{D}$  that are of constant rank. Further,  $L$  will be of the form 'kinetic energy minus potential' where the kinetic energy defines a Riemannian metric  $\mu$  on the configuration manifold.

A  $G$ -Chaplygin system is a non-holonomic system  $(Q, \mathcal{D}, L)$  which is invariant under a free and proper action by a Lie group  $G$  on  $Q$  such that  $\mathcal{D}$  defines a connection on the principal bundle  $Q \rightarrow Q/G$ . It is not required that  $\mathcal{D}$  is the mechanical connection associated to  $\mu$ . Under these assumptions the equations of motion can be written in a particularly nice format. Let  $S = Q/G$  be the reduced configuration space,  $\Omega^S$  the canonical symplectic form on  $T^*S$ ,  $J : T^*Q \rightarrow \mathfrak{g}^*$  the standard momentum map of the lifted  $G$ -action on  $T^*Q$ , and  $K \in \Omega^2(Q, \mathfrak{g})$  be the curvature form associated to  $\mathcal{D}$ . Then the non-holonomic system can be reduced, or *compressed*, to a dynamical system on  $T^*S$  with dynamics given by the vector field  $X_{\text{nh}}$  which is defined by

$$i(X_{\text{nh}})\Omega_{\text{nh}} = d\mathcal{H}_c \text{ where } \Omega_{\text{nh}} := \Omega^S - \langle J, K \rangle. \quad (1.1)$$

Here  $\mathcal{H}_c : T^*S \rightarrow \mathbb{R}$  is the *compressed Hamiltonian*; it is the function induced from the Legendre transform of  $L$ . The term  $\langle J, K \rangle$  does make sense as a semi-basic two-form on  $T^*S$  since ambiguities cancel out. The form  $\Omega_{\text{nh}}$  is, in general, an almost symplectic form, that is, it is non-degenerate and non-closed. We will thus view the compressed system  $(T^*S, \Omega_{\text{nh}}, \mathcal{H}_c)$  as an *almost Hamiltonian* system. See [18, 8, 2] and Section 2. (Our sign in (1.1) is different from that in [8] because of our choice of sign in  $\Omega^S = -d\theta$ : [8] choose  $\Omega^S = d\theta$  whence for them  $\Omega_{\text{nh}} = \Omega^S + \langle J, K \rangle$ .)

The present paper is only concerned with non-holonomic systems that arise as  $G$ -Chaplygin systems, and the description of the dynamics in terms of the above mentioned compression process will be our starting point. The question arises whether the compressed system  $(T^*S, \Omega_{\text{nh}}, \mathcal{H}_c)$  is Hamiltonizable: is there a positive function  $f : S \rightarrow \mathbb{R}$  such that  $f\Omega_{\text{nh}}$  is closed? If this is the case one says that  $\Omega_{\text{nh}}$  is conformally symplectic. The interpretation is that one is looking for an  $s \in S$  dependent time reparametrization  $d\tau = f dt$  so that the system becomes Hamiltonian in the new time. That is, the dynamics described by the vector field  $\frac{1}{f}X_{\text{nh}}(\tilde{c}(\tau)) = \frac{\partial t}{\partial \tau} \frac{\partial c}{\partial t} = \frac{\partial}{\partial \tau} \tilde{c}(\tau)$ , where  $\tilde{c}(\tau) = c(t)$ , are Hamiltonian in the usual sense with respect to  $f\Omega_{\text{nh}}$ . Moreover, it follows that the volume form  $f^{m-1}\Omega_{\text{nh}}^m$  ( $m = \dim S$ ) is preserved by the flow of  $X_{\text{nh}}$ . Conversely, when a preserved volume form  $F\Omega_{\text{nh}}^m$  exists then  $F^{\frac{1}{m-1}}$  is a candidate for a conformal factor. See the discussion in Ehlers, Koiller, Montgomery and Rios [8].

The classical Chaplygin sphere problem ([6]) is that of a dynamically balanced 3-dimensional ball that rolls on a horizontal table without slipping. Dynamically balanced means that the geometric center coincides with the center of mass. However, we do not suppose that the mass distribution is homogeneous. The inertia matrix can be any symmetric positive definite three by three matrix. The no slip condition is a non-holonomic constraint on the velocities. The ball is allowed to rotate about its vertical axis. The reduced equations were first found and integrated by Chaplygin [6] in terms of hyper-elliptic functions. A thorough study of the algebraic integrability is given in Duistermaat [7] where it is explicitly stated that the system is not Hamiltonian.

Chaplygin's rolling ball is a  $G$ -Chaplygin system with configuration space  $Q = \text{SO}(3) \times \mathbb{R}^2$ , constraint distribution  $\mathcal{D}$ , kinetic energy Lagrangian, and symmetry group  $G = \mathbb{R}^2$ . Thus  $\mathcal{D}$  defines a horizontal connection on  $Q \rightarrow S = \text{SO}(3)$ . See Section 4 for details. In [8] the compression of this system to an almost Hamiltonian system  $(T^*S, \Omega_{\text{nh}}, \mathcal{H}_c)$  is carried out. Further, [8] prove that this compressed system is not Hamiltonizable (at the  $T^*\text{SO}(3)$ -level), not even in the homogeneous case. On the other hand, Borisov and Mamaev [4, 5] give explicit formulas for a Poisson bracket which allow to write the (reduced) equations of motion for Chaplygin's ball as a true Hamiltonian system. (Their bracket is explained in geometric terms involving *affine almost Poisson structures* in [14].) Their result is all the more remarkable as it is in apparent contradiction to the assertions of [7, 8].

It actually seems to be a general phenomenon that integrable non-holonomic systems are related to integrable Hamiltonian systems. See also [12, 16, 17]. This observation provides an important motivation for a systematic study of Hamiltonization of integrable non-holonomic systems.

Chaplygin's rolling ball is the topic of Section 4. We describe its compression  $(T^*S, \Omega_{\text{nh}}, \mathcal{H}_c)$  in detail, write the system in the form (1.1), and pay particular attention to the fact that there remains a further symmetry group even after compression. Indeed, rotation of the ball about its vertical axis induces an  $S^1$ -action on the compressed phase space  $T^*S$  that preserves  $\mathcal{H}_c$  and  $\Omega_{\text{nh}}$ . In emphasizing the role of the  $\langle J, K \rangle$ -term and the almost Hamiltonian point of view we follow very closely the exposition of [8].

The main theme of the present paper is to establish a synthesis between the papers of [4, 5] and [8]. The crucial idea (actually due to [8]) which is used in this note is that the compressed system should be further reduced with respect to the induced  $S^1$ -action, and Hamiltonization should be attempted afterwards on the ultimate reduced space  $T^*S^2 = T^*(S/S^1)$ . (This is in agreement with [4, 5] since the symplectic leaves of their Poisson bracket can be realized as magnetic cotangent bundles over  $S^2$ .) The  $S^1$ -symmetries are generally referred to as internal symmetries of the system. Describing the corresponding reduction procedure is non-trivial and is the main result of the paper. (See Theorems 3.3 and 4.1.)

This problem can be stated also for higher dimensional Chaplygin balls. Let  $S = \text{SO}(n)$  be the shape space of the  $n$ -dimensional Chaplygin ball rolling on an  $n - 1$ -dimensional horizontal plane with internal symmetry group  $H = \text{SO}(n - 1)$ . Internal symmetries are very well behaved in that they give rise to conserved quantities: the standard momentum map  $J_H : T^*S \rightarrow \mathfrak{h}^*$  with respect to the canonical form  $\Omega^S$  is constant along flow lines of  $X_{\text{nh}}$ . However,  $J_H$  is *not* the momentum map with respect to  $\Omega_{\text{nh}}$ . That is, for  $\lambda \in \mathfrak{h}^*$ , the restriction of  $\Omega_{\text{nh}}$  to  $J_H^{-1}(\lambda)$  does not define a basic two form on the bundle  $J_H^{-1}(\lambda) \rightarrow J_H^{-1}(\lambda)/H_\lambda$  whence the system does not descend to the 'would be' ultimate reduced space. This is true already for  $n = 3$ . Now the point of Theorem 4.1 is that  $\Omega_{\text{nh}}$  can be *truncated* in a way that does not affect the equations of motion but does provide the correct momentum map. Effectively we replace  $\Omega_{\text{nh}}$  by a new two form  $\tilde{\Omega}$  that is non-degenerate,  $H$ -invariant, and satisfies

$$i(X_{\text{nh}})\tilde{\Omega} = d\mathcal{H}_c \text{ as well as } i(\zeta_Y)\tilde{\Omega} = \langle dJ_H, Y \rangle$$

for all  $Y \in \mathfrak{h}$  where  $\zeta_Y$  denotes the infinitesimal generator associated to  $Y$ . Why the name truncation? To construct  $\tilde{\Omega}$  we use an  $H$ -connection on the principal bundle  $T^*S \rightarrow (T^*S)/H$  such that  $X_{\text{nh}}$  is horizontal. We employ this connection to truncate the  $\langle J, K \rangle$ -term in such a way that it becomes horizontal with respect

to the  $H$ -action and we retain only the necessary information about the dynamics. In particular, Theorem 4.1 gives an explicit formula

$$\tilde{\Omega} = \Omega^S - \langle L, \text{Curv}^\omega \rangle$$

where  $\text{Curv}^\omega \in \Omega^2(S, \mathfrak{h})$  is the curvature form associated to the Hopf connection on  $\text{SO}(n) \rightarrow \text{SO}(n)/H = S^{n-1}$  and  $L : T^*S \rightarrow \mathfrak{h}^*$  is a certain mapping (related to angular velocity in the space frame) that coincides with  $J_H$  if and only if the ball is homogeneous. Notice also that  $\tilde{\Omega}$  is of the same format ‘canonical form minus semi-basic’ as  $\Omega_{\text{nh}}$ . Now, one can carry out almost Hamiltonian reduction ([20]) of  $(T^*S, \tilde{\Omega}, \mathcal{H}_c)$  with respect to the  $H$ -action.

It follows immediately, for any dimension  $n$ , that the ultimate reduced system on  $T^*S^{n-1}$  (or rather on a fiber bundle  $J_H^{-1}(\lambda)/H_\lambda \rightarrow T^*S^{n-1}$  – see Corollary 4.2) is Hamiltonian when the ball is homogeneous.

For the non-homogeneous case, thanks to the formula for  $\tilde{\Omega}$  we can reprove the result of [4] on Hamiltonization of the 3-dimensional ball in relatively simple geometric terms. See Proposition 4.4. Hamiltonization of Chaplygin’s ball for higher dimensions is still an open problem. It is, however, hoped that Theorem 4.1 can be of some help in this direction. (After this paper was finished, important progress was made by [17].)

The proof of Hamiltonization of the 3-dimensional ball given in [5] relies on Chaplygin’s reducing multiplier theorem. This theorem applies only to a certain kind of almost Hamiltonian systems with two degrees of freedom, and states that existence of a preserved measure is equivalent to existence of a conformal factor. An alternative method that has been used to prove Hamiltonization of higher dimensional non-holonomic systems is to explicitly establish an isomorphism with a classical Hamiltonian system [12, 16, 17]. Our approach is valuable in that it is purely geometric, it does not have an a-priori dimension restriction, and it ties together the work of [8] and [4, 5].

In Section 3 we study general  $G$ -Chaplygin systems with internal symmetries. In this context we describe a reduction procedure that is similar to reduction in stages in symplectic geometry. Section 4 is used as a motivation for doing so but can be read independently since all the results are proved directly. The set-up in this context is a generalization of the Chaplygin ball described above. Thus  $\pi : Q \rightarrow S$  is a  $G$ -principal fiber bundle with connection one-form  $\mathcal{A}$  and  $\mu$  is an invariant metric. The internal symmetries are modeled by *two* additional free and proper actions, called  $l$  and  $d$ , of the *same* Lie group  $H$  on  $Q$  satisfying appropriate compatibility conditions with regard to the connection  $\mathcal{A}$  and the metric  $\mu$  and the projection  $Q \rightarrow S$ . The compression of the data  $(Q, \mathcal{D} = \mathcal{A}^{-1}(0), L = \frac{1}{2} \|\cdot\|_\mu)$  and the induced  $H$ -action on  $T^*S$  are described. From the non-holonomic Noether theorem it is concluded that the standard momentum map  $J_H : T^*S \rightarrow \mathfrak{h}^*$  with respect to the canonical symplectic form on  $T^*S$  is constant along flow lines of  $X_{\text{nh}}$ . But  $J_H$  need not be the momentum map associated to  $\Omega_{\text{nh}}$ . However, we can replace  $\Omega_{\text{nh}}$  with a non-degenerate and  $H$ -invariant two form  $\tilde{\Omega}$  which not only gives the correct dynamics,  $i(X_{\text{nh}})\tilde{\Omega} = i(X_{\text{nh}})\Omega_{\text{nh}}$ , but also the desired momentum map  $J_H$ ,  $i(\zeta_Y)\tilde{\Omega} = \langle dJ_H, Y \rangle$  for all  $Y \in \mathfrak{h}$ . This is accomplished via truncation with respect to a choice of an auxiliary connection  $\sigma \in \Omega^1(T^*S, \mathfrak{h})$  on the principal bundle  $T^*S \rightarrow (T^*S)/H$ . Again  $\sigma$  is subject to the condition that  $X_{\text{nh}}$  be horizontal. Such a  $\sigma$  is shown to always exist over an open sub-manifold of  $T^*S$  which is invariant under the  $H$ -action and the dynamics of  $X_{\text{nh}}$ . This process of compression being

followed by reduction of internal symmetries has very much the flavor of reduction in stages. Indeed, when  $\mathcal{A}$  is the mechanical connection associated to  $\mu$  one can replace compression followed by reduction by usual reduction in stages.

**2. The almost Hamiltonian setting and compression.** A non-holonomic system is a triple  $(Q, \mathcal{D}, L)$  where  $Q$  is a configuration manifold,  $L : TQ \rightarrow \mathbb{R}$  is a Lagrangian, and  $\mathcal{D} \subset TQ$  is a smooth non-integrable distribution which is supposed to be of constant rank. The equations of motion for a curve  $q(t)$  which should satisfy  $q' \in \mathcal{D}$  are then stated in terms of the Lagrange d'Alembert principle. We shall only be concerned with Lagrangians of the form  $L(q, v) = \frac{1}{2}\mu_q(v, v) - V(q)$  where  $\mu$  is a Riemannian metric on  $Q$  and  $V : Q \rightarrow \mathbb{R}$  is a potential. In this case there is also an (almost) Hamiltonian version (see [2, 24], e.g.): continue to use the symbol  $\mu$  to denote the co-metric and consider the Hamiltonian  $\mathcal{H}(q, p) = \frac{1}{2}\mu(p, p) + V(q)$ . Since  $\mathcal{D}$  is of constant rank there is a family of independent one-forms  $\phi^a \in \Omega(Q)$  such that  $\mathcal{D}$  is the joint kernel of these. In terms of coordinates  $(q^i, p_i)$  the equations of motion are

$$(q^i)' = \frac{\partial \mathcal{H}}{\partial p_i} \text{ and } p_i' = -\frac{\partial \mathcal{H}}{\partial q^i} - \sum \lambda_a \phi^a \left( \frac{\partial}{\partial q^i} \right)$$

where the  $\lambda_a$  are the Lagrange multipliers to be determined from the supplementary condition that  $\mu(\phi^a, p) = 0$ . With  $X^{\mathcal{M}} := (q', p')$  we may thus rephrase the equations as

$$i(X^{\mathcal{M}})\Omega = d\mathcal{H} + \sum \lambda_a \tau^* \phi^a$$

where  $\Omega = -d\theta$  is the canonical symplectic form on  $T^*Q$  and  $\tau : T^*Q \rightarrow Q$  is the footpoint projection. (The space  $\mathcal{M} \subset T^*Q$  which is a pseudonym for the distribution  $\mathcal{D}$  will be defined below.)

Roughly speaking, the process of writing the equations of motion for a non-holonomic system in an almost Hamiltonian way amounts to eliminating the Lagrange multipliers from the equations of motion, and encoding the forces of constraint in a bracket of functions (which fails the Jacobi identity) or a (non-closed) two-form. Once this is accomplished the constraints are satisfied automatically. This process is developed below in terms of a two-form.

**2.A. Chaplygin systems.** Let  $G$  be a Lie group that acts freely, properly and by isometries on the Riemannian manifold  $(Q, \mu)$ . A  $G$ -CHAPLYGIN SYSTEM is a non-holonomic system  $(Q, L = \frac{1}{2}\|\cdot\|_{\mu}^2, \mathcal{D})$  that has the property that  $\mathcal{D}$  is a principal connection on the principal bundle  $Q \twoheadrightarrow Q/G$ . Thus  $\mathcal{D}$  is the kernel of a connection form  $\mathcal{A} : TQ \rightarrow \mathfrak{g}$ . Notice that we do not require  $\mathcal{A}$  to be the mechanical connection associated to  $\mu$ . (In principle one could also include a  $G$ -invariant function  $V : Q \rightarrow \mathbb{R}$  but we will not have use for this.)

We will now assume that  $(Q, L, \mathcal{D})$  is a  $G$ -Chaplygin system and repeat some of the constructions that are done in [2]. In fact [2] proceed in greater generality. However, in the sequel we will only be interested in Chaplygin systems whence the infinitesimal group orbit directions form an exact complement to the distribution  $\mathcal{D}$ , and this facilitates the development.

There is a sub-manifold

$$\mathcal{M} := \check{\mu}(\mathcal{D})$$

that corresponds to the constraint distribution, and the inclusion will be denoted by  $\iota : \mathcal{M} \hookrightarrow T^*Q$ . Clearly,  $\mathcal{M}$  is invariant under the cotangent lifted action by

$G$ , and there is an induced connection  $\iota^*\tau^*\mathcal{A} : T\mathcal{M} \rightarrow \mathfrak{g}$  on the principal bundle  $\mathcal{M} \twoheadrightarrow \mathcal{M}/G$ . Its horizontal space will be called

$$\mathcal{C} := (\iota^*\tau^*\mathcal{A})^{-1}(0).$$

(This corresponds to the space  $H$  in [2].)

**Theorem 2.1** ([2]). *The fiber-wise restriction of  $\iota^*\Omega$  to  $\mathcal{C}$  is non-degenerate.*

Let us denote this restriction by  $\Omega^{\mathcal{C}}$ . For the simple reason that  $\mathcal{C}$  is not the tangent space of any manifold one cannot say that  $\Omega^{\mathcal{C}}$  is a two-form. Nevertheless, morally it is this restriction process that destroys the closedness property of  $\iota^*\Omega$ . Since  $X^{\mathcal{M}}$  is tangent to  $\mathcal{M}$  and takes values in  $\mathcal{C}$  one may thus rewrite the equations of motion in the appealing format

$$i(X^{\mathcal{M}})\Omega^{\mathcal{C}} = (d\mathcal{H})^{\mathcal{C}}$$

where  $(d\mathcal{H})^{\mathcal{C}}$  is the restriction of  $d\mathcal{H}$  to  $\mathcal{C}$ .

**2.B. Compression of  $G$ -Chaplygin systems.** In this section we review the compression of  $G$ -Chaplygin systems from the Hamiltonian perspective. This will also allow us to introduce some additional notation. The original references are [18, 2, 19]. We shall follow [19] and use the word compression instead of non-holonomic reduction.

Consider a  $G$ -Chaplygin system on a configuration manifold  $Q$  with constraint distribution  $\mathcal{D} = \ker \mathcal{A}$  as defined above. Recall that  $Q$  is endowed with the kinetic energy metric  $\mu$ . Let  $\mu_0$  denote the induced metric on  $S$  that makes  $\pi$  a Riemannian submersion. (To facilitate the notation, we will sometimes tacitly identify tangent and cotangent space of  $Q$  and  $S$  via their respective metrics.) Consider the orbit projection map

$$\rho : \mathcal{M} \twoheadrightarrow \mathcal{M}/G.$$

Using the respective metrics we can write  $\rho$  as the composition

$$\rho : \mathcal{M} \cong_{\mu} \mathcal{D} \xrightarrow{T\pi|_{\mathcal{D}}} TS \cong_{\mu_0} T^*S = \mathcal{M}/G. \quad (2.2)$$

We may also associate a fiber-wise inverse to this mapping which is given by the horizontal lift mapping  $\text{hl}^{\mathcal{A}}$  associated to  $\mathcal{A}$ . (This inverse was called the clock-wise diagram in [8, Section 3.1].) As already noted above,  $\tilde{\mathcal{A}} := \iota^*\tau^*\mathcal{A} : T\mathcal{M} \rightarrow \mathfrak{g}$  defines a principal bundle connection for  $\rho$ , whose horizontal spaces are given by  $\mathcal{C}$ . (The connection  $\tilde{\mathcal{A}}$  is the same as the one obtained in [8] by differentiating the clock-wise diagram.)

**Proposition 2.2** (Compression). *The following are true.*

- (1)  $\Omega^{\mathcal{C}}$  descends to a non-degenerate two-form  $\Omega_{\text{nh}}$  on  $T^*S$ .
- (2)  $\Omega_{\text{nh}} = \Omega_S - \langle J_G \circ \text{hl}^{\mathcal{A}}, \tau_S^* K \rangle$ . Here  $\Omega_S = -d\theta_S$  is the canonical form on  $T^*S$ ,  $J_G$  is the momentum map of the cotangent lifted  $G$ -action on  $T^*Q$ ,  $K \in \Omega^2(S, \mathfrak{g})$  is the curvature form of  $\mathcal{A}$ , and  $\tau_S : T^*S \rightarrow S$  is the projection.
- (3) The vector field  $X^{\mathcal{M}}$  is  $\rho$ -related to the vector field  $X_{\text{nh}}$  on  $T^*S$  defined by  $i(X_{\text{nh}})\Omega_{\text{nh}} = d\mathcal{H}_c$  where the compressed Hamiltonian,  $\mathcal{H}_c : T^*S = TS \rightarrow \mathbb{R}$  is defined by  $\mathcal{H}_c := \mathcal{H} \circ \text{hl}^{\mathcal{A}}$ , with  $\text{hl}^{\mathcal{A}}$  denoting the horizontal lift mapping.

This result is well-known. It is contained in [2, 18, 19, 8]. Nevertheless we include the following proof of the ‘ $\langle J, K \rangle$ -formula’ because it is slightly different in its flavor from those that can be found in the literature. Note that our sign in the ‘ $\langle J, K \rangle$ -formula’ differs from that in [8] since we are using a different convention for

the canonical exact symplectic form. (For sake of brevity we will sometimes write  $\langle J_G, K \rangle$  instead of  $\langle J_G \circ \text{hl}^A, \tau_S^* K \rangle$ .)

*Proof.* We need to show that  $\Omega^{\mathcal{C}} = \rho^*(\Omega^S - \langle J_G \circ \text{hl}^A, \tau_S^* K \rangle)$  on  $\mathcal{C}$ . Work locally, i.e., assume that  $Q = S \times G$  is a direct product and that  $TS = S \times U$  is trivializable. Via right trivialization we shall also identify  $TG = G \times \mathfrak{g}$ . Thus the connection is given by

$$\mathcal{A} : S \times U \times G \times \mathfrak{g} \longrightarrow \mathfrak{g}, (s, u, g, v) \longmapsto v + \mathcal{A}_{(s,g)}(u). \quad (2.3)$$

The horizontal space is  $\mathcal{M} = \mathcal{D} = \{(s, u, g, -\mathcal{A}_{(s,g)}(u))\}$ . Let  $X_1, X_2$  be vector-fields on  $\mathcal{M}$  with values in  $\mathcal{C} = \tilde{\mathcal{A}}^{-1}(0)$  that project to vector-fields  $\bar{X}_1, \bar{X}_2 \in \mathfrak{X}(T^*S)$ . Since  $X_i \in \mathcal{C}$  we have  $X_i = (s'_i, u'_i, -\mathcal{A}(s'_i), v'_i)$  for  $i = 1, 2$ . Note also that  $\bar{X}_i = (s'_i, u'_i)$  and  $T(\tau \circ \iota).X_i = (s'_i, -\mathcal{A}(s'_i)) \in T(S \times G)$ .

Let  $\psi : T\mathcal{M} \rightarrow \mathcal{C}$  denote the horizontal projection associated to  $\tilde{\mathcal{A}}$ .

Claim:  $\theta^Q \circ \psi = \rho^* \theta^S$  where  $\theta^Q, \theta^S$  denote the respective canonical one-forms. Indeed, for  $(q, p) \in \mathcal{M}$  and  $\rho(q, p) = \rho(s, g, u, -\mathcal{A}_{(s,g)}(u)) = (s, u)$  we find

$$\theta^Q X_2(q, p) = \mu_q((u, -\mathcal{A}(u)), (s'_2, -\mathcal{A}(s'_2))) = (\mu_0)_s(u, s'_2) = \theta^S \bar{X}_2(s, u)$$

which shows the identity for all vector-fields where it is non-trivial. Therefore,

$$X_1.\theta^Q X_2 = d((\theta^S \bar{X}_2) \circ \rho).X_1 = (d(\theta^S \bar{X}_2).d\rho(X_1)) \circ \rho = (\bar{X}_1.\theta^S \bar{X}_2) \circ \rho.$$

Since  $\psi + \zeta \circ \tilde{\mathcal{A}} = \text{id}_{T\mathcal{M}}$  where  $\zeta$  is the fundamental vector-field mapping of the  $G$ -action on  $\mathcal{M}$ , it follows that

$$\begin{aligned} \Omega^{\mathcal{M}}(X_1, X_2) &= -X_1.\theta^Q X_2 + X_2.\theta^Q X_1 + \theta^Q \psi[X_1, X_2] + \theta^Q(\zeta \circ \tilde{\mathcal{A}})[X_1, X_2] \\ &= -\rho^*(\bar{X}_1.\theta^S \bar{X}_2 - \bar{X}_2.\theta^S \bar{X}_1) + \rho^*(\theta^S[\bar{X}_1, \bar{X}_2]) - \langle J_G, \tau^* K \rangle(\bar{X}_1, \bar{X}_2) \\ &= -\rho^*(d\theta^S)(X_1, X_2) - \rho^*\langle J_G \circ \text{hl}^A, \tau_S^* K \rangle(X_1, X_2). \end{aligned}$$

For the middle equation we used the following identity. Let  $\tilde{K} \in \Omega^2(\mathcal{M}, \mathfrak{g})$  denote the curvature form associated to  $\tilde{\mathcal{A}}$ . Then  $\theta^Q(\zeta \circ \tilde{\mathcal{A}})[X_1, X_2] = \langle J_G, \tilde{\mathcal{A}}[X_1, X_2] \rangle = \langle J_G, -\tilde{K}(X_1, X_2) \rangle = \langle J_G, -K \circ \Lambda^2 T\tau(\bar{X}_1, \bar{X}_2) \rangle$ .  $\square$

We collect the compressed data to a triple  $(T^*S, \Omega_{\text{nh}}, \mathcal{H}_c)$  and refer to it as the compressed system. Let us also note explicitly that the equations of motion are now given by the almost Hamiltonian form

$$i(X_{\text{nh}})\Omega_{\text{nh}} = d\mathcal{H}_c \quad (2.4)$$

whence the constraints have been successfully encoded in the two-form structure. In general,  $\Omega_{\text{nh}}$  is an almost symplectic form, that is, it is non-degenerate and non-closed. Thus we refer to the compressed system  $(T^*S, \Omega_{\text{nh}}, \mathcal{H}_c)$  as an almost Hamiltonian system. However, there are non-integrable distributions which do give rise to forms  $\Omega_{\text{nh}}$  that are closed. This is simply so because compression is a generalization of symplectic reduction at the 0-level of simple mechanical systems. Consider, e.g., the homogeneous Veselova system of [25]. For this system the configuration space  $Q$  is  $\text{SO}(3)$ , the group  $G$  is  $S^1$ , and the distribution  $\mathcal{D}$  is the horizontal space of the mechanical connection associated to the standard biinvariant metric on  $\text{SO}(3)$ . (The constraints are conserved quantities of the unconstrained system.) Thus compression and symplectic reduction at  $0 \in \mathfrak{g}^* = \mathbb{R}$  agree.

**3. Reduction of internal symmetries via truncation.** We continue notation and assumptions from Section 2.B. Thus  $\pi : Q \rightarrow S$  is a  $G$ -principal fiber bundle with connection form  $\mathcal{A}$ . Additionally we assume that there is a Lie group  $H$  which acts on  $S$ , through a linear representation on  $\mathfrak{g}$ , and by two different actions,  $l$  and  $d$ , on  $Q$ . More precisely we require that

- $\pi : Q \rightarrow S$  is  $l$ - and  $d$ -equivariant;
- $\mathcal{A} : TQ \rightarrow \mathfrak{g}$  is  $d$ -equivariant;
- $l$  acts by internal symmetries, that is  $\mathcal{A} \cdot \zeta_Y^l = 0$  for all  $Y \in \mathfrak{h}$ .

The metric  $\mu$  on  $Q$  is now supposed to be  $l$ -,  $d$ -, and  $G$ -invariant. This is the abstraction of the situation encountered in Section 4.

In non-holonomic mechanics the relationship between symmetries and conserved quantities is not obvious. (See [3].) While the momentum map for an external symmetry group (the  $G$ - and  $d$ -actions) is generally not constant during the motion, the momentum map associated to a internal symmetry (the  $l$ -action) is. This is the non-holonomic version of Noether's theorem which we state for further reference in the following theorem that can be found in [1].

**Theorem 3.1.** *Let  $H$  be an internal symmetry group of a non-holonomic system. Then the momentum map  $J_H : T^*Q \rightarrow \mathfrak{h}^*$  is constant during the motion.*

By an internal symmetry of  $(Q, \mathcal{D}, L)$  we mean an action by a Lie group  $H$  on  $Q$  such that  $L$  is  $H$ -invariant and  $\zeta_Y \in \mathcal{D}$  for all  $Y \in \mathfrak{h}$ . However,  $\mathcal{D}$  is not required to be  $H$ -invariant.

**3.A. Compression in the presence of internal symmetries.** Via the metric we identify  $TQ$  and  $T^*Q$  and the horizontal bundle  $\mathcal{D} = \mathcal{A}^{-1}(0)$  is identified with its image  $\mathcal{M} \subset T^*Q$ . Let  $\mu_0$  denote the induced metric on  $S$ . As in Section 2.B we denote the compressed Hamiltonian by  $\mathcal{H}_c := \mathcal{H} \circ \text{hl}^A$  where  $\text{hl}^A : TS \rightarrow \mathcal{D}$  is the horizontal lift mapping. Recall also the projection  $\rho : \mathcal{M} \rightarrow \mathcal{M}/G = T^*S$ . The following describes how the internal symmetries descend to the compressed system  $(T^*S, \Omega_{\text{nh}}, \mathcal{H}_c)$ .

**Proposition 3.2.** *The following are true.*

- (1) *The  $H$ -action  $d$  restricts to  $\mathcal{M}$ , and  $\rho$  is equivariant with respect to the cotangent lifted  $H$ -action on  $T^*S$ .*
- (2)  *$\Omega_{\text{nh}}$  is  $H$ -invariant.*
- (3)  *$\mathcal{H}_c$  and  $X_{\text{nh}}$  are  $H$ -invariant.*
- (4)  *$J_H = (J_l|_{\mathcal{M}}) \circ \text{hl}^A$  where  $J_H$  is the standard momentum map of the cotangent lifted  $H$ -action on  $(T^*S, \Omega^S)$  and  $J_l$  is the standard momentum map of the lifted  $l$ -action on  $(T^*Q, \Omega^Q)$ .*
- (5)  *$dJ_H \cdot X_{\text{nh}} = 0$ .*

Note that  $l$  does not necessarily restrict to an action on  $\mathcal{M}$  and  $J_d$  (the momentum map of the  $d$ -action) does not factor to  $J_H$ .

*Proof.* (1) This is clear from the assumptions.

(2) Note that  $\Omega^S$  is clearly invariant. Further,  $J_G$  is  $H$ -equivariant with respect to the  $d$ -action since  $\zeta^G : \mathfrak{g} \rightarrow TQ$  is  $d$ -equivariant by assumption, and the same is true for  $\text{hl}^A$  and  $K$ . That is,  $h^* \langle J_G \circ \text{hl}^A, K \rangle = \langle h^*(J_G \circ \text{hl}^A), h.K \rangle = \langle J_G \circ \text{hl}^A, K \rangle$  for all  $h \in H$ , since  $h^* \alpha = \alpha \circ h^{-1}$  for  $\alpha \in \mathfrak{g}^*$ .

(3) Since  $\text{hl}^A$  is  $H$ -equivariant for the  $d$ -action and  $\mathcal{H}$  is invariant the first point is clear. For the second we use that also  $\Omega_{\text{nh}}$  is  $H$ -invariant whence

$$h^*X_{\text{nh}} = h^*((\check{\Omega}_{\text{nh}})^{-1}d\mathcal{H}_c) = ((h^*\Omega_{\text{nh}})^{\check{}})^{-1}h^*d\mathcal{H}_c = X_{\text{nh}}$$

for  $h \in H$ .

(4) Let  $Y \in \mathfrak{h}$  and  $(s, u) \in T^*S = TS$ , then  $\text{hl}^A_q(\zeta_Y(s)) = \zeta_Y^l(q)$  implies that

$$\langle J_H(s, u), Y \rangle = (\mu_0)_s(u, \zeta_Y(s)) = \mu_q(\text{hl}^A_q(u), \zeta_Y^l(q)) = \langle J_l(\text{hl}^A(s, u)), Y \rangle.$$

(5) This follows from the previous point and Theorem 3.1.  $\square$

**3.B. Truncation.** By the above proposition the compressed system  $(T^*S, \Omega_{\text{nh}}, X_{\text{nh}})$  is  $H$ -invariant and  $J_H$  is a conserved quantity. Thus one would be tempted to do almost Hamiltonian reduction. (See [20].) However,  $J_H$  is not the momentum map corresponding to  $\Omega_{\text{nh}}$ , that is, for  $Y \in \mathfrak{h}$

$$i(\zeta_Y)\Omega_{\text{nh}} \neq d\langle J_H, Y \rangle$$

in general. Thus the restriction of  $\Omega_{\text{nh}}$  to a level set of  $J_H$  will not be a horizontal form in general whence it does not factor to a reduced form on the ‘would be’ almost symplectic quotient. (This is the situation for the Chaplygin ball problem treated in Section 4.) To remedy the situation we truncate the  $\langle J_G, K \rangle$ -term thus changing  $\Omega_{\text{nh}}$  in a certain way that does not affect the equations of motion. In effect, we will replace  $(T^*S, \Omega_{\text{nh}}, \mathcal{H}_c)$  by a different almost Hamiltonian system  $(T^*S, \tilde{\Omega}, \tilde{\mathcal{H}}_c)$  which has the same dynamics given by  $X_{\text{nh}}$ .

To motivate the construction notice that the obstruction to horizontality of the restriction of  $\Omega_{\text{nh}}$  to a level set of  $J_H$  is just

$$i(\zeta_Y)\Omega_{\text{nh}} - d\langle J_H, Y \rangle = i(\zeta_Y)\Omega_{\text{nh}} - i(\zeta_Y)\Omega^S = -i(\zeta_Y)\langle J, K \rangle.$$

So the vertical directions are problematic. On the other hand, we have for the dynamics

$$d\mathcal{H}_c = i(X_{\text{nh}})\Omega^S - i(X_{\text{nh}})\langle J, K \rangle.$$

Thus we need to invent a device whereby we make the  $\langle J, K \rangle$ -term vanish upon insertion of vertical vectors while it remains unchanged when contracted with  $X_{\text{nh}}$ . In particular we need a way to distinguish  $X_{\text{nh}}$  from vertical directions. This calls for a connection such that  $X_{\text{nh}}$  is horizontal.

**Theorem 3.3.** *Suppose there is a connection  $\sigma \in \Omega(T^*S, \mathfrak{h})$  of the principal bundle  $T^*S \rightarrow (T^*S)/H$  that satisfies  $\sigma X_{\text{nh}} = 0$ . (See Proposition 3.4.) Let  $\chi : TT^*S \rightarrow TT^*S$  denote the horizontal projection associated to  $\sigma$ . Then the truncated form*

$$\tilde{\Omega} := \Omega^S - \langle J_G \circ \text{hl}^A, \tau_S^*K \rangle \circ \Lambda^2\chi$$

has the following properties.

- (1) It is non-degenerate.
- (2) It is  $H$ -invariant.
- (3)  $i(X_{\text{nh}})\tilde{\Omega} = d\mathcal{H}_c$ .
- (4)  $i(\zeta_Y)\tilde{\Omega} = d\langle J_H, Y \rangle$  for all  $Y \in \mathfrak{h}$ .

*Proof.* Properties (1) and (2) are immediate. (Use that  $\chi$  is  $H$ -equivariant for the second.)

(3) We need to show that  $\tilde{\Omega}(X_{\text{nh}}, X) = \Omega_{\text{nh}}(X_{\text{nh}}, X)$  for all  $X \in \mathfrak{X}(T^*S)$ . If  $X$  is horizontal then this is obvious. Suppose  $X$  is vertical, that is  $X = \zeta_Y$  for some  $Y \in \mathfrak{h}$ . But then we have

$$\langle J_G, K \rangle(X_{\text{nh}}, \zeta_Y) = 0;$$

this follows because

$$\Omega_{\text{nh}}(X_{\text{nh}}, \zeta_Y) = d\mathcal{H}_c.\zeta_Y = 0$$

by  $H$ -invariance of  $\mathcal{H}_c$ , and

$$\Omega^S(X_{\text{nh}}, \zeta_Y) = -\langle dJ_H.X_{\text{nh}}, Y \rangle = 0$$

by conservation of  $J_H$ . Thus  $\tilde{\Omega}(X_{\text{nh}}, \zeta_Y) = \Omega^S(X_{\text{nh}}, \zeta_Y) = \Omega_{\text{nh}}(X_{\text{nh}}, \zeta_Y)$ .

(4) This is true since the truncated  $\langle J_G, K \rangle$ -term vanishes, by construction, on vertical vectors and  $J_H$  is the canonical momentum map.  $\square$

Note that the above proof relies on both decisive features of an almost Hamiltonian system with symmetries: it uses invariance of the Hamiltonian as well as the conserved quantity.

When  $\mathcal{A}$  is the mechanical connection on  $Q \rightarrow S$  associated to the metric  $\mu$  then compression equals symplectic reduction at 0. Thus  $\Omega_{\text{nh}} = \Omega^S$  is a true symplectic form in this case and the  $H$ -action is Hamiltonian with momentum map  $J_H$ . Obviously, this is compatible with the truncation procedure in the trivial sense. Thus we recover symplectic reduction in stages.

Of course, there may also be a connection  $\tilde{\sigma} \in \Omega^1(Q, \mathfrak{h})$  with the property that  $X_{\text{nh}} \in \ker \tau^*\tilde{\sigma}$ . If this is the case then one can replace  $\sigma$  in the theorem by  $\tau^*\tilde{\sigma}$  but in general this seems to be too much to ask for. The analog of Proposition 3.4 does not hold.

Thus to describe the dynamics of  $X_{\text{nh}}$  we may deal with the system  $(T^*S, \tilde{\Omega}, \mathcal{H}_c)$  which has the advantage that it not only admits  $H$  as a symmetry group but also produces the desired momentum map. Now one can perform Hamiltonian reduction ([20]) with respect to the non-closed form  $\tilde{\Omega}$ .

Now we address the question of existence of the auxiliary connection  $\sigma$  needed for truncation. Consider the vertical space  $\text{Ver}(H)$  of the lifted  $H$ -action on  $T^*S$ . Define the sets

$$\mathcal{E} = X_{\text{nh}}^{-1}(\text{Ver}(H)) \text{ and } \mathcal{U} = (T^*S) \setminus \mathcal{E}$$

and note that  $\mathcal{U}$  is an open sub-manifold while  $\mathcal{E}$  is the set of relative equilibria.

**Proposition 3.4.** *The following are true.*

- (1)  $\mathcal{U}$  and  $\mathcal{E}$  are  $H$ -invariant and invariant under the dynamics of  $X_{\text{nh}}$ .
- (2) On  $\mathcal{U}$  there is a connection  $\sigma$  such that  $X_{\text{nh}}$  is horizontal.

*Proof.* (1) It suffices to show the assertions for  $\mathcal{E}$ . Invariance under the  $H$ -action is clear since  $X_{\text{nh}}$  and  $\text{Ver}(H)$  are  $H$ -invariant. Fix  $(s, u) \in \mathcal{E}$  such that  $X_{\text{nh}}(s, u) = \zeta_Y(s, u)$  for some  $Y \in \mathfrak{h}$ , and consider the curve  $c(t) = \exp(tY).(s, u)$ . By  $H$ -invariance the curve stays in  $\mathcal{E}$ . We show that it is an integral curve:

$$c'(t) = \zeta_Y(\exp(tY).(s, u)) = \exp(tY).\zeta_Y(s, u) = \exp(tY).X_{\text{nh}}(s, u) = X_{\text{nh}}(c(t)).$$

(2) Consider the  $H$ -invariant sub-bundle of  $T\mathcal{U}$  given by  $\mathcal{F} = \mathbb{R}X_{\text{nh}} \oplus \text{Ver}(H)$  where  $\text{Ver}(H)$  is the vertical space of the induced  $H$ -action on  $\mathcal{U}$ . Take an  $H$ -invariant metric on  $\mathcal{U}$ . Such a metric always exists since the action is proper. Now define a horizontal sub-bundle by  $\text{Hor}(\sigma) = \mathbb{R}X_{\text{nh}} \oplus \mathcal{F}^\perp$  where the orthogonal is taken with respect to the metric. By construction  $\text{Hor}(\sigma)$  is an  $H$ -invariant complement of the vertical space of the  $H$ -action on  $\mathcal{U}$ .  $\square$

Consider a point  $(s, u) \in \mathcal{E}$ . Clearly  $s' = T\tau_S.X_{\text{nh}}(s, u) = u$  where we identify again  $T^*S = TS$  via  $\mu_0$ . Since  $(s, u) \in \mathcal{E}$  there is a  $Y \in \mathfrak{h}$  such that  $X_{\text{nh}}(s, u) = \zeta_Y^{TS}(s, u)$  whence  $u = \zeta_Y^l(s)$ . In the physical applications we have in mind it is true that  $\mu(\zeta_Y^l, \zeta_v^G) = 0$ . Therefore,  $\langle J_G(s, u), v \rangle = \mu_q(\text{hl}^A(u), \zeta_v^G) = -\mu_q(\zeta_Y^l, \zeta_v^G) = 0$  whence the  $\langle J_G, K \rangle$ -term vanishes upon restriction to  $\mathcal{E}$ . We can thus understand the dynamics on  $(T^*S, \Omega_{\text{nh}}, \mathcal{H}_c)$  by treating  $(\mathcal{E}, \Omega^S|_{\mathcal{E}}, \mathcal{H}_c|_{\mathcal{E}})$  and  $(\mathcal{U}, \tilde{\Omega}, \mathcal{H}_c|_{\mathcal{U}})$  as individual problems.

**4. Example: Chaplygin's rolling ball.** The  $n$ -dimensional Chaplygin ball ( $n \geq 3$ ) concerns a rigid ball that rolls on an  $n - 1$ -dimensional table without slipping and whose geometric center coincides with its center of mass. The mass distribution is not assumed to be homogeneous.

By adjusting the units appropriately we assume the radius and the mass of the ball both equal to 1. It is convenient to identify the table with  $\mathbb{R}^{n-1} \times \{-1\}$  whence the motion of the center of the ball is given by a curve  $(x(t), 0) \in \mathbb{R}^{n-1} \times \{0\}$ .<sup>1</sup> Let  $e_1, \dots, e_n$  denote the standard basis of  $\mathbb{R}^n$ . The orientation of the ball at time  $t_0$  is determined by a unique element  $s(t_0) \in \text{SO}(n) =: S$  that relates this basis to a moving frame that is attached to the center of the ball and rotates with it. Thus the configuration space of the system is

$$Q := S \times \mathbb{R}^{n-1}.$$

Consider a fixed marked point  $b$  on the surface of the ball. The motion of this point is described by the curve

$$z(t) = (x(t), 0) + s(t).b.$$

The constraint of rolling without slipping is that the velocity of the contact point is 0. For the contact point at time  $t_0$  we have that  $s(t_0).b = -e_n$  whence  $z'(t_0) = 0$  implies that

$$(x'(t_0), 0) = s'(t_0)s(t_0)^{-1}.e_n.$$

We put  $s'(t_0)s(t_0)^{-1} = \tilde{u} \in \mathfrak{so}(n)_{\mathbb{R}}$  where  $\mathfrak{so}(n)_{\mathbb{R}}$  is identified with the Lie algebra of right invariant vector fields on  $S$ . In other words, the constraints are satisfied iff

$$(s' s^{-1}, x') \in \tilde{\mathcal{D}} := \{(\tilde{u}, x') \in \mathfrak{so}(n)_{\mathbb{R}} \times \mathbb{R}^{n-1} : \tilde{u}.e_n = (x', 0)\}.$$

Thus the set of allowed motions is described by the condition that the velocities in the right trivialization (space frame) belong to  $\tilde{\mathcal{D}}$ . If we define

$$\tilde{\mathcal{A}} : \mathfrak{so}(n)_{\mathbb{R}} \longrightarrow \mathbb{R}^{n-1}, \quad \tilde{u} \longmapsto \tilde{u}.e_n \longmapsto - \sum_{a=1}^{n-1} \langle e_a, \tilde{u}.e_n \rangle_E e_a,$$

where  $\langle \cdot, \cdot \rangle_E$  denotes the standard inner product, then

$$\tilde{\mathcal{D}} = \{(s, \tilde{u}, x, -\tilde{\mathcal{A}}(\tilde{u}))\} \subset S \times \mathfrak{so}(n)_{\mathbb{R}} \times T\mathbb{R}^{n-1}.$$

The sign in the definition of  $\tilde{\mathcal{A}}$  is included so that the associated horizontal subspace (see below) can be written in the usual way. Let

$$H := \{h \in S : h.e_n = e_n\}$$

with Lie algebra  $\mathfrak{h}$ , let  $\langle \cdot, \cdot \rangle$  denote the Killing form, and let  $\mathfrak{h} \oplus \mathfrak{h}^\perp$  be the corresponding decomposition of  $\mathfrak{so}(n)$ . When appropriate we will identify  $H = \text{SO}(n-1)$  and consider it as acting on  $\mathbb{R}^{n-1}$ . In terms of matrix notation  $\mathfrak{h}^\perp$  corresponds to

<sup>1</sup>We will write row vectors but treat them as column vectors.

the subspace of matrices that have non-zero entries only in the last column and row. Let

$$Y_\alpha, Z_a, \alpha = 1, \dots, \dim \mathfrak{h}, a = 1, \dots, n-1$$

denote an orthonormal basis that is adapted to this decomposition. Then, if the basis is ordered and oriented in the right way, we may write  $\tilde{\mathcal{A}} = -\sum_a \langle Z_a, \cdot \rangle e_a$ . It will be convenient to work with the left trivialization (body frame). From now on we trivialize  $TS = S \times \mathfrak{so}(n)$  via the left trivialization. Consider the  $\mathbb{R}^{n-1}$ -valued one-form on  $S$  defined by

$$\mathcal{A} : TS = S \times \mathfrak{so}(n) \longrightarrow \mathbb{R}^{n-1}, (s, u) \longmapsto \text{Ad}(s).u = \tilde{u} \longmapsto \tilde{\mathcal{A}}(\tilde{u}).$$

Via right multiplication we extend the basis  $Y_\alpha, Z_a$  to a frame on  $S$ :

$$\xi_\alpha(s) := \zeta_{Y_\alpha}^l(s) = \text{Ad}(s^{-1})Y_\alpha \text{ and } \zeta_a(s) := \zeta_{Z_a}^l(s) = \text{Ad}(s^{-1})Z_a. \quad (4.5)$$

The corresponding co-frame will be called  $\rho^\alpha, \eta^a$ . We shall stick to the convention of using lower case Greek letters  $\alpha, \beta, \gamma$  to refer to  $Y_\alpha$ 's and lower Latins  $a, b, c$  for  $Z_a$ 's. In this frame the form  $\mathcal{A} \in \Omega^1(S, \mathbb{R}^{n-1})$  reads

$$\mathcal{A} = -\sum_{a=1}^{n-1} \eta^a e_a.$$

For  $n = 3$  one can get the same formula for  $\mathcal{A}$  as in [8] by inserting appropriate signs which corresponds to rearranging the basis. All such choices cancel out in the subsequent.

Let  $\mathbb{I}$  denote the inertia tensor that describes the mass distribution of the ball. Then the appropriate metric on  $Q$  is the product metric  $\mu = \langle \mathbb{I}, \cdot \rangle + \langle \cdot, \cdot \rangle_E$  and the Lagrangian of the system is the kinetic energy function associated to  $\mu$ . Thus the Chaplygin ball is the non-holonomic system described by the triple

$$(Q, \mathcal{D}, \frac{1}{2} \|\cdot\|_\mu) \quad (4.6)$$

where  $\mathcal{D}$  is the sub-bundle defined by

$$\mathcal{D} = \{(s, u, x, -\mathcal{A}_s(u))\} \subset S \times \mathfrak{so}(n) \times T\mathbb{R}^{n-1}.$$

Notice that the kinetic energy of the system is left invariant while the distribution that describes the constraints is right invariant. The system (4.6) is invariant under the Lie group action given by addition of  $\mathbb{R}^{n-1}$  on the  $\mathbb{R}^{n-1}$ -factor of  $Q$ . Clearly,  $\mathcal{D}$  defines a connection on the principal bundle  $\mathbb{R}^{n-1} \hookrightarrow Q \rightarrow S$  with connection form  $\mathcal{A} \in \Omega^1(S, \mathbb{R}^{n-1})$ . Thus (4.6) is a  $G$ -Chaplygin system in the sense of Section 2 with  $G = \mathbb{R}^{n-1}$ . Moreover,  $\mathcal{A}$  has the following properties.

- (1)  $\mathcal{A}(hs, u) = h\mathcal{A}(s, u)$  for all  $h \in H$  and  $(s, u) \in TS$ , that is,  $\mathcal{D}$  is invariant under the diagonal  $H$ -action.
- (2)  $H$  acts by internal symmetries, that is,  $\mathcal{A}.\zeta_Y^l = 0$  for all  $Y \in \mathfrak{h}$ .
- (3)  $\mathcal{A}(sg^{-1}, \text{Ad}(g)u) = \mathcal{A}(s, u)$  for all  $g \in S$  and  $(s, u) \in TS$ .

These properties have a physical meaning. Property (1) says that the constraints are invariant under simultaneous rotation of the space frame and the ball about the vertical axis. The second says that rotation of the ball about the vertical axis is an allowed motion. The third states that the system is invariant with respect to rotations of the space frame. Notice also that properties (1) and (2) correspond to the compatibility conditions stated at the beginning of Section 3.

4.A. **The compressed system.** As stated above the Lagrangian  $\mathcal{L}$  of the system is the kinetic energy associated to  $\mu$ . This Lagrangian is non-degenerate whence we are in the situation of Section 2, and we will denote the corresponding Hamiltonian by  $\mathcal{H}$ . For ease of notation we will write  $V := \mathbb{R}^{n-1}$ .

Let  $\Omega^Q$  be the canonical symplectic form on  $T^*Q$ . In accordance with Section 2.B we describe now the compression of the system  $(Q, \mathcal{L}, \mathcal{D} = \{(s, u; x, -\mathcal{A}_s(u))\})$ . We will henceforth identify

$$T^*S = TS = \mathcal{D}/V \quad (4.7)$$

via the induced metric  $\mu_0$ . The compressed Hamiltonian reads

$$\mathcal{H}_c(s, u) = \frac{1}{2}\langle u, \mathbb{I}u \rangle + \frac{1}{2}\langle \mathcal{A}_s(u), \mathcal{A}_s(u) \rangle,$$

and note that  $\mathcal{H}_c$  is invariant under the induced  $H$ -action on  $TS$ . (This action has various equivalent descriptions – see Proposition 3.2.) The Hamiltonian  $\mathcal{H}_c$  is the sum of a left- and a right-invariant factor. Systems of this type are sometimes called  $L + R$ -systems. See [11].

According to Section 2.B the compressed almost symplectic form on  $TS$  is of the form

$$\Omega_{\text{nh}} = \Omega^S - \langle J_V \circ \text{hl}^A, d\mathcal{A} \rangle = \Omega^S + \langle \mathcal{A}, d\mathcal{A} \rangle.$$

It will be convenient to introduce the following set of functions on  $TS$ :

$$l_\alpha(s, u) = \rho_s^\alpha(u), \quad \tilde{l}_\alpha(s, u) = l_\alpha(s, \mathbb{I}u), \quad \text{and} \quad g_a(s, u) = \eta_s^a(u), \quad \tilde{g}_a(s, u) = g_a(s, \mathbb{I}u)$$

where  $\rho^\alpha, \eta^a$  denotes the co-frame associated to (4.5). These functions have a physical meaning;  $l_\alpha, g_a$  are the components of angular velocity in the space frame and  $\tilde{l}_\alpha, \tilde{g}_a$  are those of angular momentum about the center of mass also in the space frame. We may thus write the canonical symplectic form as

$$\Omega^S = -d\left(\sum \tilde{l}_\alpha \rho^\alpha + \sum (\tilde{g}_a + g_a) \eta^a\right). \quad (4.8)$$

(Remember that the identification of  $TS$  with its dual is via  $\mu_0$ .) The formulas

$$d\rho^\alpha = \frac{1}{2} \sum c_{\beta\gamma}^\alpha \rho^\beta \wedge \rho^\gamma + \frac{1}{2} \sum c_{ab}^\alpha \eta^a \wedge \eta^b \quad \text{and} \quad d\eta^a = \sum c_{\beta b}^a \rho^\beta \wedge \eta^b \quad (4.9)$$

will be used very often; here the summation is over repeated indices and  $c_{..}$  are the structure constants. The compressed form thus becomes

$$\Omega_{\text{nh}} = \Omega^S + \sum g_a c_{\beta b}^a \rho^\beta \wedge \eta^b. \quad (4.10)$$

Via the trivialization we write the non-holonomic vector-field  $X_{\text{nh}} = (\Omega_{\text{nh}}^\vee)^{-1} d\mathcal{H}$  on  $TS$  as

$$X_{\text{nh}}(s, u) = (s'(s, u), u'(s, u)) \in \mathfrak{so}(n) \times \mathfrak{so}(n).$$

Using right invariant vector fields we thus have that

$$s' = \sum l_\alpha \xi_\alpha + \sum g_a \zeta_a. \quad (4.11)$$

This is just the first half of Hamilton's equations which says that  $s' = u$ .

According to Proposition 3.2 the momentum map associated to the  $l$ -action compresses to the standard momentum map

$$J_H : TS \longrightarrow \mathfrak{h}^* =_{\langle \cdot, \cdot \rangle} \mathfrak{h}, \quad (s, u) \longmapsto \sum \tilde{l}_\alpha(s, u) Y_\alpha$$

with respect to the lifted  $H$ -action on  $(TS, \Omega^S)$ . Furthermore, we have the conservation law  $dJ_H \cdot X_{\text{nh}} = 0$ .

**4.B. Truncation.** We are now in the situation of Section 3.B. Namely one can verify that the conserved quantity  $J_H$  is *not* the momentum map with respect to  $\Omega_{\text{nh}}$ . Thus  $\Omega_{\text{nh}}$  does *not* factor to a two form on quotients of the type  $J_H^{-1}(\lambda)/H\lambda$ . Therefore, we need to change  $\Omega_{\text{nh}}$  in a certain way.

According to Theorem 3.3 we have to find a connection  $\sigma$  on the principal bundle  $TS \rightarrow (TS)/H$  such that  $X_{\text{nh}}$  is horizontal. This means that  $\chi(X_{\text{nh}}) = X_{\text{nh}}$  where  $\chi : T(TS) \rightarrow T(TS)$  is the associated horizontal projection. Let us also trivialize

$$T(TS) = T(S \times \mathfrak{so}(n)) = TS \times T\mathfrak{so}(n) = S \times \mathfrak{so}(n) \times \mathfrak{so}(n) \times \mathfrak{so}(n)$$

via left-multiplication. Then  $\sigma$  has to be of the form

$$\sigma = \left( \sum (\rho^\alpha + f_a^\alpha \eta^a) \otimes \xi_\alpha, 0 \right)$$

where the  $f_a^\alpha = f_a^\alpha(s, u)$  are unknown functions. Thus

$$\chi = \left( - \sum f_a^\alpha \eta^a \otimes \xi_\alpha + \sum \eta^a \otimes \xi_a, \text{id}_{\mathfrak{so}(n)} \right).$$

The condition that  $X_{\text{nh}}$  be horizontal becomes

$$l_\alpha = - \sum f_a^\alpha g_a. \quad (4.12)$$

In accordance with Proposition 3.4 this is solvable on the complement of the set

$$\mathcal{E} = (X_{\text{nh}})^{-1}(\mathfrak{h} \times \{0\}) = \{g_a = 0\} \subset TS.$$

However, for convenience of exposition we restrict to the somewhat smaller set  $\mathcal{U}' := \{(s, u) : g_a(s, u) \neq 0\} \subset \mathcal{E}^c$ . One particular choice for  $\chi$  that solves equation (4.12) is

$$\chi = \left( \frac{1}{n-1} \sum \frac{l_\alpha}{g_a} \eta^a \otimes \xi_\alpha + \sum \eta^a \otimes \xi_a, \text{id}_{\mathfrak{so}(n)} \right).$$

(In fact, since  $\chi$  has to be  $H$ -equivariant one does not have so much freedom here. Choosing  $f_a^\alpha = -\frac{l_\alpha}{g_a} \delta_{1a}$  solves (4.12) but does not yield an equivariant  $\chi$ , for example.)

Our strategy will now be to truncate  $\langle J, K \rangle$  using  $\chi$ . This truncation will be well-defined on  $\mathcal{U}'$  only. However, it will be obvious how to extend the result to a two-form on the whole space.

When  $n = 3$  the truncated form is especially easy to compute. (For notational reasons we make the convention that  $a = 1, 2$  and  $\alpha = 3$  whence the basis receives the appellation  $Z_1, Z_2, Y_3$ .) Indeed,

$$\langle J, K \rangle(\chi\zeta_1, \chi\zeta_2) = - \sum g_a c_{ab}^\alpha \rho^\alpha \wedge \eta^b (\chi\zeta_1, \chi\zeta_2) = l_3 = \frac{1}{2} \sum c_{ab}^\alpha l_\alpha \eta^a \wedge \eta^b (\zeta_1, \zeta_2).$$

Thus we can replace  $\langle J, K \rangle$  with the semi-basic two form

$$\widetilde{\langle J, K \rangle} := \frac{1}{2} \sum c_{ab}^\alpha l_\alpha \eta^a \wedge \eta^b$$

which is obviously well-defined on the whole space and also makes sense for  $n > 3$ .

However, notice that  $\widetilde{\langle J, K \rangle} \neq \langle J, K \rangle \circ \Lambda^2 \chi$  for  $n > 3$ . They agree only on a set of measure zero and the truncated form is not defined on the whole space. The point is that the contraction with  $X_{\text{nh}}$  does not see this difference whence we may use  $\widetilde{\langle J, K \rangle}$ .

Now we notice that  $\widetilde{\langle J, K \rangle}$  can be written in terms of well known geometric objects. Namely, let

$$\tilde{\Omega} := \Omega^S - \widetilde{\langle J, K \rangle} = \Omega^S - \langle L, \text{Curv}^\omega \rangle$$

where  $L = \sum l_\alpha Y_\alpha$  and  $\text{Curv}^\omega \in \Omega^2(S, \mathfrak{h})$  is the curvature of the standard  $H$ -connection  $\omega = \sum \rho^\alpha \otimes Y_\alpha$ . Thus  $L$  is the one-form  $\omega$  viewed as a function  $TS \rightarrow \mathfrak{h}$ .

**Theorem 4.1.** *The system  $(TS, \tilde{\Omega}, \mathcal{H}_c)$  has the following properties.*

- (1)  $\tilde{\Omega}$  is almost symplectic and  $H$ -invariant;
- (2)  $i(X_{\text{nh}})\tilde{\Omega} = d\mathcal{H}_c$ ;
- (3)  $J_H$  is a momentum map of the  $H$ -action on  $(TS, \tilde{\Omega})$ .

*Proof.* Clearly  $\tilde{\Omega}$  is non-degenerate, and the term  $\langle L, \text{Curv}^\omega \rangle$  is  $H$ -invariant because ambiguities in the pairing cancel out. The third assertion is also obvious since

$$i(\xi_\alpha)\tilde{\Omega} = i(\xi_\alpha)\Omega^S = \langle dJ_H, Y_\alpha \rangle.$$

Thus it remains to show that

$$i(X_{\text{nh}})\langle J, K \rangle = i(X_{\text{nh}})\langle L, \text{Curv}^\omega \rangle.$$

Notice that  $\langle J, K \rangle(X_{\text{nh}}, \xi_\alpha) = 0$  by the proof of Theorem 3.3. Equating on  $\omega$ -horizontal vector fields and using formula (4.11) for  $T\tau.X_{\text{nh}}$  yields

$$\langle J, K \rangle(X_{\text{nh}}, \zeta_c) = - \sum g_a c_{\alpha b}^a \rho^\alpha \wedge \eta^b(X_{\text{nh}}, \zeta_c) = - \sum c_{ac}^a g_a l_\alpha.$$

On the other hand:

$$\langle L, \text{Curv}^\omega \rangle(X_{\text{nh}}, \zeta_c) = \sum_{a < b} l_\alpha c_{ab}^\alpha \eta^a \wedge \eta^b(X_{\text{nh}}, \zeta_c) = \sum c_{ac}^\alpha g_a l_\alpha$$

where we have used that

$$\begin{aligned} \text{Curv}^\omega &= d\omega - \frac{1}{2}[\omega, \omega] = \sum d\rho^\alpha Y_\alpha - \sum_{\alpha < \beta} \rho^\alpha \wedge \rho^\beta c_{\alpha\beta}^\gamma Y_\gamma \\ &= \sum_{\beta < \gamma, b < c} (c_{\beta\gamma}^\alpha \rho^\beta \wedge \rho^\gamma + c_{bc}^\alpha \eta^b \wedge \eta^c) Y_\alpha - \sum_{\alpha < \beta} \rho^\alpha \wedge \rho^\beta c_{\alpha\beta}^\gamma Y_\gamma \\ &= \sum_{b < c} c_{bc}^\alpha \eta^b \wedge \eta^c Y_\alpha \end{aligned}$$

which follows from formulas (4.9).  $\square$

The theorem thus provides a particular choice of a truncating two-form. When  $n = 3$  this is the only possible choice. Indeed, this is so because a two-form in three dimensions is already fixed by specifying its contractions (to one-forms) with respect to two transversal vector fields. The two vector fields are  $X_{\text{nh}}$  and the infinitesimal generator of the  $H$ -action. Of course, one is really only interested in the point-wise tangent projections of these vector fields. Indeed, to tie this to [14] notice that the two-form  $-i(X_{\text{nh}})\nu$  defined in [14] is just  $\langle J, K \rangle - \langle L, \text{Curv}^\omega \rangle$ ; the form  $\nu = \rho^1 \wedge \eta^1 \wedge \eta^2$  is the standard volume form on  $S = \text{SO}(3)$ .

In higher dimensions, however, there will be many different possibilities, and it is not clear whether these are all on an equal footing. For example, are there choices which yield a form  $\tilde{\Omega}$  which becomes (conformally) closed after restriction to a level set of  $J_H$  while this is not true for other choices?

The existence of  $\tilde{\Omega}$  in the above proposition allows to replace the triple  $(TS, \Omega_{\text{nh}}, \mathcal{H})$  with the triple  $(TS, \tilde{\Omega}, \mathcal{H})$ . This leaves the dynamics unaltered but has the advantage that the conserved quantity  $J_H$  is now the momentum map associated to the  $H$ -symmetry. We can thus do (almost) Hamiltonian reduction and pass to the quotient  $J_H^{-1}(\mathcal{O})/H$  where  $\mathcal{O} \subset \mathfrak{h}^*$  is a coadjoint orbit.

**Corollary 4.2** (The ultimate reduced phase space). *Let  $\mathcal{O} \subset \mathfrak{h}^*$  be a coadjoint orbit. Then*

$$J_H^{-1}(\mathcal{O})/H \cong TS^{n-1} \times_{S^{n-1}} (S \times_H \mathcal{O})$$

where the isomorphism depends on the mechanical connection on  $S \rightarrow S/H$  associated to the metric  $\mu_0$ . In particular,  $J_H^{-1}(\mathcal{O})/H$  is isomorphic to a bundle over  $TS^{n-1}$  with fiber  $\mathcal{O}$ .

*Proof.* This follows from the usual argument involving the mechanical connection and the locked inertia tensor associated to  $\mu_0$ .  $\square$

Let  $\lambda \in \mathcal{O}$ . Since  $H \times_{H_\lambda} J_H^{-1}(\lambda) \cong J_H^{-1}(\mathcal{O})$  where  $H_\lambda$  is the stabilizer subgroup at  $\lambda$  we can also do point reduction to arrive at the same reduced space, that is,  $J_H^{-1}(\lambda)/H_\lambda = J_H^{-1}(\mathcal{O})/H$ . This implies the following corollary.

**Corollary 4.3.** *When  $\mathbb{I} = 1$  Chaplygin's ball is Hamiltonian after reduction by  $H$ .*

*Proof.* In this case  $L = J_H$  and closedness follows from the Bianchi identity for the curvature form.  $\square$

We stress that truncation is necessary even in the homogeneous case. This is due to the fact that  $\mathcal{D}$  is never the horizontal space of the mechanical connection associated to  $\mu$ . Once the non-holonomic two-form  $\Omega_{\text{nh}}$  has been altered one can perform reduction and it is only then that the system becomes Hamiltonian. This should be compared with [8, Section 3.3]. See also the remarks in Section 5.

**4.C. Hamiltonization of the 3-dimensional ball.** Let  $n = 3$ . Consider the metric isomorphism  $\Phi := (\mu_0)^\vee = \mathbb{I} + \mathcal{A}^* \mathcal{A} : TS \rightarrow T^*S =_{\langle \cdot, \cdot \rangle} TS$ ,  $(s, u) \mapsto \mathbb{I}u + \sum g_a(s, u) \text{Ad}(s^{-1})Z_a$ . Define

$$f(s) = (\det \Phi_s)^{-\frac{1}{2}} \text{ where } s \in S.$$

Because of  $H$ -invariance  $f$  drops to a function  $S^2 \rightarrow \mathbb{R}$ .<sup>2</sup>

**Proposition 4.4** (Hamiltonization). *Let  $\lambda \in \mathfrak{h}^* \cong \mathbb{R}$ . Then  $d(f\tilde{\Omega})|_{J_H^{-1}(\lambda)} = 0$ .*

*Proof.* Let  $\iota : J_H^{-1}(\lambda) \hookrightarrow TS$  be the inclusion. Notice that

$$\begin{aligned} \iota^* d(f\tilde{\Omega}) &= \iota^*(df \wedge \Omega^S - df \wedge \langle L, \text{Curv}^\omega \rangle - f d\langle L, \text{Curv}^\omega \rangle) = 0 \\ &\iff df \wedge \theta^S - f \langle L, \text{Curv}^\omega \rangle \text{ is closed on } J_H^{-1}(\lambda). \end{aligned}$$

Since  $\mu_0$  is  $H$ -invariant it follows that  $\xi_\alpha \cdot \Phi = 0$  and using the derivation property of the determinant function we find that

$$df = -\frac{1}{2} (\det \Phi)^{-\frac{3}{2}} \det(\Phi) \sum \text{Tr}(\Phi^{-1} \zeta_a \cdot \Phi) \eta^a = -\frac{1}{2} f \sum \text{Tr}(\Phi^{-1} \zeta_a \cdot \Phi) \eta^a.$$

Computing the trace with respect to the orthonormal basis  $\text{Ad}(s^{-1})Y_\alpha, \text{Ad}(s^{-1})Z_a$  gives

$$N_a := \text{Tr}(\Phi^{-1} \zeta_a \cdot \Phi) = -2 \sum \langle \Phi^{-1} c_{ab}^\alpha \text{Ad}(s^{-1})Y_\alpha, \text{Ad}(s^{-1})Z_b \rangle.$$

<sup>2</sup>This function was called  $\rho_\mu$  in [5, Section 3] and has also been considered in [10] in the context of higher dimensional Chaplygin systems.

Actually  $\alpha = 1$  and  $a = 1, 2$  because  $n = 3$ . However, for notational reasons we will make the convention that  $\alpha = 3$ . The basis of  $\mathfrak{so}(3)$  is thus called  $Z_1, Z_2, Y_3$ . Therefore,

$$\begin{aligned} & df \wedge \theta^S - f \langle L, \text{Curv}^\omega \rangle \\ &= f \left( -\frac{1}{2} \sum N_a \eta^a \wedge (\tilde{l}_3 \rho^3 + (\tilde{g}_b + g_b) \eta^b) - l_3 \eta^1 \wedge \eta^2 \right) \\ &= -f \left( \frac{1}{2} \sum N_a \tilde{l}_3 \eta^a \wedge \rho^3 + \left( \frac{1}{2} (N_1(\tilde{g}_2 + g_2) - N_2(\tilde{g}_1 + g_1)) + l_3 \right) \eta^1 \wedge \eta^2 \right) \end{aligned}$$

Notice that that the first term in this expression,  $-\frac{1}{2} f \sum N_a \eta^a \wedge \tilde{l}_3 \rho^3 = df \wedge \tilde{l}_3 \rho^3$ , becomes closed upon restriction to a level set of  $J_H = \tilde{l}_3 Y_3$ . For the middle term, a short calculation using that  $n = 3$  now shows that

$$N_1(\tilde{g}_2 + g_2) - N_2(\tilde{g}_1 + g_1) = -2l_3 - 2\langle \Phi^{-1} \text{Ad}(s^{-1}) Y_3, \text{Ad}(s^{-1}) Y_3 \rangle \tilde{l}_3.$$

Therefore,

$$f \left( \frac{1}{2} (N_1(\tilde{g}_2 + g_2) - N_2(\tilde{g}_1 + g_1)) + l_3 \right) \eta^1 \wedge \eta^2 = -f \langle \Phi^{-1} \text{Ad}(s^{-1}) Y_3, \text{Ad}(s^{-1}) Y_3 \rangle \tilde{l}_3 \eta^1 \wedge \eta^2$$

which is also closed when restricted to a level set of  $J_H = \tilde{l}_3 Y_3$ .  $\square$

This approach gives a symplecto-geometric explanation of the formulas in [4, 5]. Note in particular that the proof involves rather little computation.

Unfortunately the above proof relies very heavily on the fact that  $n = 3$ . However, it is designed so that, in principle, all the expressions also make sense in higher dimensions. It is hoped that this approach can also be useful in studying cases of Hamiltonization in dimensions  $n > 3$ . Indeed, it would be very nice if these techniques could be used to give a useful characterization of those inertia matrices  $\mathbb{I}$  and values of  $J_H$  which yield a system that is Hamiltonizable after reduction by  $H$ .

**5. Comments and conclusions.** One of the goals of this paper was to work out the reduction of  $G$ -Chaplygin systems with respect to additional internal symmetries modeled by a Lie group  $H$  subject to the compatibility conditions described in Section 3. The first step was to describe the compression to an almost Hamiltonian system  $(T^*S, \Omega_{\text{nh}}, \mathcal{H}_c)$  in the presence of internal symmetries. A construction that is similar to this step can also be found in [22, 21]. The novelty in the truncation procedure is that we can reduce the dynamics of the system to a coadjoint bundle over  $T^*B = T^*(S/H)$  and *reproduce* the structure of an almost Hamiltonian system. This gives a general answer to a question posed for the special case of the 3-dimensional Chaplygin ball problem in [9, Section 4.1].

The main technical step in our reduction procedure is called *truncation*. This involves a choice of a principal bundle connection  $\sigma$  on  $T^*S \rightarrow (T^*S)/H$  such that the non-holonomic vector field  $X_{\text{nh}}$  is horizontal. The name is chosen because, effectively, we use the connection  $\sigma$  to cut off all the information contained in the  $\langle J, K \rangle$ -term that is not seen by the dynamics but presents an obstruction to reduction.

In Section 4 we apply this reduction procedure to the  $n$ -dimensional Chaplygin ball problem. Thus we write the system as an almost Hamiltonian system on a coadjoint bundle over  $T^*(S^{n-1})$ . In particular we derive a symplectic proof of the remarkable result of [4, 5] on the Hamiltonizability of the 3-dimensional Chaplygin ball.

Furthermore, we can also deal with the  $n$ -dimensional homogeneous Chaplygin ball. In Corollary 4.3 we show that this system is Hamiltonian after reduction of

internal symmetries (but not at the compressed level). From the mathematical point of view this is a non-trivial conclusion: even in the homogeneous case the connection  $\mathcal{D}$  does not coincide with the mechanical connection associated to  $\mu$ , whence one cannot employ usual symplectic reduction techniques to construct the reduced phase space. In fact, [8] have shown (for  $n = 3$ ) that the problem is not even Hamiltonizable (i.e., conformally symplectic) at the compressed level. Thus one has to use truncation to eliminate the internal symmetries, and it is only then that the system becomes Hamiltonian. On the other hand, the result is obvious from a physical perspective: Consider the big phase space  $T^*Q = T^*(S \times \mathbb{R}^{n-1})$  and the Hamiltonian  $\mathcal{H}$  of the ball. Let  $X_{\mathcal{H}}$  denote the Hamiltonian vector field associated to  $\mathcal{H}$  with respect to the *canonical symplectic* structure on  $T^*Q$ . This is the homogeneous  $n$ -dimensional ball that rolls on a horizontal table without constraints. If this ball happens to satisfy the no-slip condition at *one* time instant it will also have to satisfy the constraints for all future and past time; it cannot accelerate and will roll on a straight line. The point is that this physical fact cannot be described in the framework of existing reduction theories: either one does symplectic reduction of the free system but then one cannot describe the constraints, i.e., the space  $\mathcal{D}$  (which could be viewed as necessary initial conditions), within this process; or one does compression which captures the constraint space  $\mathcal{D}$  but destroys the Hamiltonian feature of the system. Hence the need for truncation. See Corollary 4.3.

The truncation of  $\Omega_{\text{nh}}$  is an example of a more general procedure in which one consistently replaces the almost Hamiltonian system  $(T^*S, \Omega_{\text{nh}}, \mathcal{H}_c)$  by  $(T^*S, \tilde{\Omega}, \mathcal{H}_c)$ . Even though both systems define the same vector field on  $T^*S$ , there may an advantage in working with  $\tilde{\Omega}$ . (For instance, one may be conformally symplectic while the other is not.) This is the idea of adding an *affine term* to  $\Omega_{\text{nh}}$  which seems to go back to [23], has been formalized in [8], and successfully used in [14]. An affine term is a semi-basic two form on  $T^*S$  which vanishes when contracted with  $X_{\text{nh}}$ . The problem is how to choose the affine term. In the special case of internal symmetries the situation is easier as the symmetries provide extra information. Notice that in Section 4.B we used the truncation to find our choice of affine term. In Theorem 4.1, however, we did not use the truncated two-form  $\langle J, K \rangle \circ \Lambda^2 \chi$ , but rather another form that we found to be more convenient. Thus it is important to remember that one has many different possibilities here and the truncation is just a means to find one particular choice. More generally, the idea of modifying  $\Omega_{\text{nh}}$  seems to be important also for systems without internal symmetries (such as the rubber ball) but a systematic treatment is not known. The Dirac reduction techniques (which do not use internal symmetries) developed in [15] could provide a starting point, but it seems to us that one encounters the same difficulties as in compression.

The study of other non-holonomic systems, including the rubber ball, with this perspective is work in progress.

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