An (almost) symplectic view of Chaplygin's ball

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G-Chaplygin systems

A *non-holonomic system* is a triple $(Q, L, \mathcal{D})$ where

- Q is an n-dimensional configuration manifold;
- $\mathcal{D} \subset TQ$ is a smooth non-integrable distribution of constant rank;
- $L = E_{\text{kin}} - E_{\text{pot}} : TQ \to \mathbb{R}$ is the Lagrangian, and we assume that E_{kin} defines a Riemannian metric μ on Q.

The equations of motion for a curve $q(t)$ in Q, such that $q' \in \mathcal{D}$, are determined by the Lagrange-d'Alembert principle.

A *G-Chaplygin system* is a non-holonomic system $(Q, L, \mathcal{D})$ acted upon by a Lie group G in a free and proper fashion such that $\mathcal{D}$ defines a connection on the principal bundle

$$G \hookrightarrow Q \twoheadrightarrow Q/G =: S.$$

(Also called the principal or purely kinematical case.) In particular, G acts by isometries on (Q, μ).

We do not require that $\mathcal{D}$ is the mechanical connection.

The almost Hamiltonian formulation

Via the metric μ we can associate a Hamiltonian $\mathcal{H}$ to the Lagrangian,

$$\mathcal{H}(q,p) = \tfrac{1}{2}\mu_q(p,p) + V(q).$$

Let $\phi^a \in \Omega^1(Q)$, $a = 1, \ldots, \dim G$ be the components of the connection form such that $\mathcal{D} = \ker(\phi^a)_a$. In terms of local coordinates (q^i, p_i) on Q the equations of motion are given by

$$X^{\mathcal{M}}(q,p) := \begin{pmatrix} q^i \\ p_i \end{pmatrix}' = \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial p_i} \\ -\frac{\partial \mathcal{H}}{\partial q^i} - \sum_a \lambda_a \phi^a(\frac{\partial}{\partial q^i}) \end{pmatrix}$$

where the λ_a are the Lagrange multipliers determined from the supplementary equations $\mu(\phi^a, p) = \phi^a(v) = 0$. Thus $X^{\mathcal{M}}$ is tangent to

$$\mathcal{M} := \check{\mu}(\mathcal{D}).$$

Intrinsically this writes as

$$i(X^{\mathcal{M}})\Omega^Q = d\mathcal{H} + \sum_a \lambda_a \tau^* \phi^a$$

where $\tau : T^*Q \to Q$.

The Bates-Śniatycki construction

Let $\iota : \mathcal{M} \hookrightarrow T^*Q$ and $\mathcal{A} := (\phi^a)_a : TQ \to \mathfrak{g}$. The G-action induces an action on $\mathcal{M}$ and there is an induced connection $\iota^*\tau^*\mathcal{A}$ on the principal fiber bundle

$$G \hookrightarrow \mathcal{M} \twoheadrightarrow \mathcal{M}/G = T^*S.$$

Let

$$\mathcal{C} := (\iota^*\tau^*\mathcal{A})^{-1}(0) \subset T\mathcal{M}$$

be the horizontal space. Define $\Omega^{\mathcal{C}}$ to be the fiber-wise restriction to $\mathcal{C}$ of $\iota^*\Omega^Q$, and $(d\mathcal{H})^{\mathcal{C}}$ the restriction of $d\mathcal{H}$ to $\mathcal{C}$. Then

$$i(X^{\mathcal{M}})\Omega^{\mathcal{C}} = (d\mathcal{H})^{\mathcal{C}}.$$

Theorem ([BS93])

$\Omega^{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is non-degenerate.

Thus we can completely describe the dynamics in terms of the triple $(\mathcal{M}, \Omega^{\mathcal{C}}, \iota^*\mathcal{H})$ together with the above equation.

Can we reproduce this structure on T^*S?

Beside: The mechanical case

What happens when $\mathcal{A} : TQ \to \mathfrak{g}$ is the mechanical connection?

Then $\mathcal{M} = J^{-1}(0)$ where $J : T^*Q \to \mathfrak{g}^*$ is the standard momentum map of the lifted G-action. Further, $\mathcal{C}$ is the associated horizontal space of $\mathcal{M} \twoheadrightarrow \mathcal{M}/G = T^*S$, and non-degeneracy of $\Omega^{\mathcal{C}}$ follows from the momentum map equation

$$i(\zeta_X)\Omega^Q = d\langle J, X\rangle \text{ for all } X \in \mathfrak{g}$$

which implies that $\mathrm{Ver}(\mathcal{M} \twoheadrightarrow \mathcal{M}/G) = \ker \iota^*\Omega^Q$.

Therefore, when $\mathcal{A}$ is mechanical,

$$X^{\mathcal{M}} = X_{\mathcal{H}} = \check{\Omega}^{-1}(d\mathcal{H}) \in \mathcal{C} \subset TT^*Q|\mathcal{M}$$

since $X_{\mathcal{H}}$ is tangent to $\mathcal{M}$ by Noether's Theorem.

When $\mathcal{A}$ is *not* the mechanical connection, $\mathcal{M} \neq J^{-1}(0)$, and the dynamics of $X^{\mathcal{M}}$ will, in general, differ from those of $X_{\mathcal{H}}$.

Compression

Let $\rho : \mathcal{M} \twoheadrightarrow \mathcal{M}/G = T^*S$ denote the projection. This map has a fiber-wise inverse:

$$\mathsf{hl}^{\mathcal{A}} : T^*S \cong_{\mu_0} TS \xrightarrow{\mathsf{hl}^{\mathcal{A}}} \mathcal{D} \cong_{\mu} \mathcal{M}$$

where μ_0 is the induced metric on S and $\mathsf{hl}^{\mathcal{A}}$ is the horizontal lift.

Proposition ([BS93, K92])

- *$\Omega^{\mathcal{C}}$ descends to a non-degenerate two-form Ω_{nh} on T^*S.*
- *$\Omega_{\mathsf{nh}} = \Omega^S - \langle J_G \circ \mathsf{hl}^{\mathcal{A}}, \mathsf{Curv}_0^{\mathcal{A}} \rangle$ where $\mathsf{Curv}_0^{\mathcal{A}}$ is the induced curvature form on S.*
- *$\iota^*\mathcal{H}$ drops to a function $\mathcal{H}_{\mathsf{c}}$ on T^*S.*
- *The vector field $X^{\mathcal{M}}$ is ρ-related to the vector field $X_{\mathsf{nh}} := (\check{\Omega}_{\mathsf{nh}})^{-1} d\mathcal{H}_{\mathsf{c}}$.*

The form Ω_{nh} is, in general, not closed. The non-holonomic system $(Q, L, \mathcal{D})$ is thus encoded in the *almost* Hamiltonian system $(T^*S, \Omega_{\mathsf{nh}}, \mathcal{H}_{\mathsf{c}})$.

How far is Ω_{nh} from closedness?

Let $\chi : T\mathcal{M} \to \mathcal{C}$ denote the projection onto the horizontal subspace. Then, alternatively to $\Omega^{\mathcal{C}}$, one may also consider

$$\iota^*\Omega^Q \circ \Lambda^2\chi = -d(\iota^*\theta^Q) \circ \Lambda^2\chi = -d_{\mathcal{A}}\iota^*\theta^Q$$

which is the covariant derivative of $\iota^*\theta^Q$. (This is the extension of $\Omega^{\mathcal{C}}$ to $\Lambda^2 T\mathcal{M}$ by 0.)

Now, Ω_{nh} is closed iff $\rho^*\Omega_{\mathrm{nh}}$ is closed. But,

$$\begin{aligned} d\rho^*\Omega_{\mathrm{nh}} &= d_{\mathcal{A}}\rho^*\Omega_{\mathrm{nh}} = d_{\mathcal{A}}(\iota^*\Omega^Q \circ \Lambda^2\chi) \\ &= -d_{\mathcal{A}}^2\iota^*\theta^Q. \end{aligned}$$

Thus there are two scenarios which yield closedness of Ω_{nh}:

- The connection $\mathcal{A}$ is mechanical whence $\iota^*\Omega^Q$ is a horizontal form, that is, $\iota^*\Omega^Q \circ \Lambda^2\chi = \iota^*\Omega^Q$.
- The distribution $\mathcal{D}$ is *integrable* which means that the constraints are *holonomic*. For, $d_{\mathcal{A}}^2 = \chi^* \circ i(R) \circ d$ where $R = \zeta^G \circ \mathrm{Curv}^{\mathcal{A}}$ is the curvature.

Chaplygin systems with internal symmetries

We now furnish the G-Chaplygin system $(Q, L, \mathcal{D})$ with additional symmetries.

Suppose a Lie group H acts properly and freely by *two different* actions, l and d, on Q such that both factor to *one* H-action on S such that:

- $\pi : Q \twoheadrightarrow Q/G = S$ is l- and d-equivariant.

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- $\mathcal{A} : TQ \to \mathfrak{g}$ is d-equivariant with respect to a representation of H on $\mathfrak{g}$. (This means that $\mathcal{D}$ is d-invariant whence d defines an *external* symmetry.)

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- $\mathcal{A}.\zeta^l_Y = 0$ whence l defines an *internal* symmetry.

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- Additionally, L is l- and d-invariant.

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- $\mathcal{A}.\zeta^l_Y = 0$ whence l defines an *internal* symmetry.
- Additionally, L is l- and d-invariant.

Conservation of internal symmetries

Since l is internal it is true that $dJ^l.X^{\mathcal{M}} = 0$ where J^l is the standard momentum map associated to the l-action on T^*Q. (Non-holonomic Noether Theorem)

Compression of internal symmetries

How do these symmetries behave with respect to compression?

Proposition

- *d induces an action on $\mathcal{M}$ and $\rho : \mathcal{M} \to T^*S$ is d-equivariant.*
- *Ω_{nh} and $\mathcal{H}_{\mathrm{c}}$ are H-invariant. (Uses d.)*
- *The standard momentum map $J_H : T^*S \to \mathfrak{h}^*$ satisfies $J_H = (J_I|\mathcal{M}) \circ \mathrm{hl}^{\mathcal{A}}$.*
- *Thus, $dJ_H.X_{\mathrm{nh}} = 0$.*

I does not act on $\mathcal{M}$ and J_d does not factor to J_H.

Thus H is a symmetry group of the compressed system $(T^*S, \Omega_{\mathrm{nh}}, \mathcal{H}_{\mathrm{c}})$ and J_H is a conserved quantity.

What about reduction by H? Can we reproduce this structure on $J_H^{-1}(\lambda)/H_\lambda$ where $\lambda \in \mathfrak{h}^*$?

Reduction of internal symmetries?

Problem

J_H is not the momentum map of Ω_{nh}, i.e.,

$$i(\zeta_Y)\Omega_{\mathrm{nh}} \neq d\langle J_H, Y\rangle$$

for general $Y \in \mathfrak{h}$. That is, $\Omega_{\mathrm{nh}}|J_H^{-1}(\lambda)$ is not horizontal with respect to $J_H^{-1}(\lambda) \twoheadrightarrow J_H^{-1}(\lambda)/H_\lambda$.

Solution

Replace Ω_{nh} by $\widetilde{\Omega}$ such that:

(1) $\widetilde{\Omega}$ is non-degenerate.

(2) $i(X_{\mathrm{nh}})\widetilde{\Omega} = d\mathcal{H}_{\mathrm{c}}$.

(3) $\widetilde{\Omega}$ is H-invariant.

(4) $i(\zeta_Y)\widetilde{\Omega} = d\langle J_H, Y\rangle$ for all $Y \in \mathfrak{h}$.

Does such an $\widetilde{\Omega}$ exist?

Reduction via truncation

Theorem

Suppose there is a connection $\sigma \in \Omega^1(T^*S, \mathfrak{h})$ *on the principal bundle* $T^*S \twoheadrightarrow (T^*S)/H$ *such that* X_{nh} *is horizontal. Then the truncated form*

$$\widetilde{\Omega} := \Omega^S - \langle J_G \circ \mathrm{hl}^{\mathcal{A}}, \mathrm{Curv}_0^{\mathcal{A}} \rangle \circ \Lambda^2 \chi$$

satisfies (1)-(4); χ *is the horizontal projection associated to* σ*.*

Proof.

(1), (3), (4) are immediate.

For (2) we need to show that $\Omega_{\mathrm{nh}}(X_{\mathrm{nh}}, \xi) = \widetilde{\Omega}(X_{\mathrm{nh}}, \xi)$ for all ξ. When ξ is horizontal this is obvious. Assume that $\xi = \zeta_Y$. Then:

$$\begin{aligned} 0 &= d\mathcal{H}_{\mathrm{c}}.\zeta_Y = \Omega_{\mathrm{nh}}(X_{\mathrm{nh}}, \zeta_Y) \\ 0 &= -\langle dJ_H.X_{\mathrm{nh}}, Y \rangle = \Omega^S(X_{\mathrm{nh}}, \zeta_Y) = \widetilde{\Omega}(X_{\mathrm{nh}}, \zeta_Y). \end{aligned}$$

In particular, $\langle J_G \circ \mathrm{hl}^{\mathcal{A}}, \mathrm{Curv}_0^{\mathcal{A}} \rangle(X_{\mathrm{nh}}, \zeta_Y) = 0$.

Reduction via truncation: Existence

Let $\mathcal{E} := {X_{\mathsf{nh}}}^{-1}(\mathrm{Ver}(H))$, and $\mathcal{U} := (T^*S) \setminus \mathcal{E}$ which is open.

Theorem

- *$\mathcal{U}$ and $\mathcal{E}$ are both H- and X_{nh}-invariant.*
- *On the principal bundle $\mathcal{U} \twoheadrightarrow \mathcal{U}/H$ there is a connection σ such that X_{nh} is horizontal.*

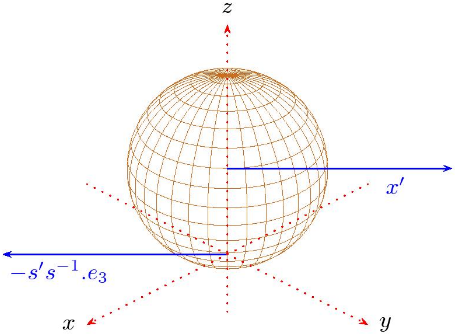
Conclusion

We may replace $(T^*S, \Omega_{\mathsf{nh}}, \mathcal{H}_{\mathsf{c}})$ by treating two separate problems: Firstly, the problem on $\mathcal{E}$ which are the relative equilibria; secondly we can do *almost Hamiltonian reduction* with respect to the H-action of $(\mathcal{U}, \widetilde{\Omega}_{\mathcal{U}}, \mathcal{H}_{\mathsf{c}}|\mathcal{U})$.

This shows that the form Ω_{nh} is 'not optimal' for the description of Chaplygin systems with internal symmetries. It carries 'too much information'.

Part II: Chaplygin's rolling ball

The Chaplygin ball is a round ball whose center of mass lies at the geometric center and which rolls without slipping or sliding on a horizontal table.



Constraints

The horizontal component of the contact point's angular velocity equals minus the ball's linear velocity. Or: The contact point has zero total velocity.

The n-dimensional Chaplygin ball

The configuration space of the n-dimensional Chaplygin ball is

$$Q := S \times V := \mathrm{SO}(n) \times \mathbb{R}^{n-1}.$$

A curve $(s(t), x(t))$ is an allowed motion iff

$$(s' s^{-1}, x') \in \widetilde{\mathcal{D}} := \{(\widetilde{u}, x') \in \mathfrak{so}(n)_R \times V : \widetilde{u}.e_n = (x', 0)^t\}.$$

Here $\mathfrak{so}(n)_R$ is the Lie algebra of *right* invariant vector fields on S. In terms of the *left* trivialization, $TS = S \times \mathfrak{so}(n)$, the Hamiltonian reads

$$\mathcal{H} = \tfrac{1}{2}\langle \mathbb{I}u, u\rangle + \tfrac{1}{2}\langle x', x'\rangle_V$$

where $u = s^{-1}s'$, $\langle ., .\rangle$ is minus the Killing form on $\mathfrak{so}(n)$ and $\mathbb{I}$ is the inertia matrix. Thus the metric is

$$\mu = \langle \mathbb{I}., .\rangle + \langle ., .\rangle_V.$$

(Units such that mass and radius equal to 1.)

Left vs. Right

Let $\widetilde{\mathcal{A}} : S \times \mathfrak{so}(n)_R \to V$, $(s, \widetilde{u}) \mapsto -\mathrm{pr}_V(\widetilde{u}.e_n)$. Then

$$\widetilde{\mathcal{D}} = \{(s, \widetilde{u}, x, -\widetilde{\mathcal{A}}(\widetilde{u}))\}.$$

In the *left* trivialization this becomes

$$\begin{aligned} \mathcal{A} : TS = S \times \mathfrak{so}(n) &\longrightarrow S \times \mathfrak{so}(n)_R \longrightarrow V \\ (s, u) &\longmapsto (s, \mathrm{Ad}(s)u) \longmapsto -\mathrm{pr}_V\big((\mathrm{Ad}(s)u).e_n\big), \end{aligned}$$

and $\widetilde{\mathcal{D}}$ translates to

$$\mathcal{D} = \{(s, u, x, -\mathcal{A}_s(u))\} \subset TS \times TV = TQ$$

which is a (*right* invariant) non-integrable distribution on Q.

The Chaplygin ball is the non-holonomic system described by the triple $(Q, \mathcal{D}, \frac{1}{2}||\cdot||^2_\mu)$. What about symmetries?

Symmetries

$(Q, \mathcal{D}, \frac{1}{2}||\cdot||_{\mu}^{2})$ is a G-Chaplygin system with $G = V$.

(1) $\mathcal{A} : TS \to V$ is a connection form on the principal fiber bundle $V \hookrightarrow Q \twoheadrightarrow S$.

(2) $\mathcal{A}$ is right invariant.

The group $H = \{h \in S : h.e_n = e_n\} = \mathrm{SO}(n-1)$ acts through *two* different actions on Q:

(3) The l-action: $l_h(s, x) = (hs, x)$. This action generates *internal* symmetries: $\mathcal{A}\zeta_Y^l = \widetilde{\mathcal{A}}Y = 0$ for all $Y \in \mathfrak{h}$.

(4) The d-action: $d_h(s, x) = (hs, hx)$. This action generates *external* symmetries. $\mathcal{A}(hs, u) = h.\mathcal{A}(s, u)$ for all $h \in H$. Thus $\mathcal{D}$ is invariant under the d-action.

Notice that $\mathcal{A}$ is never the mechanical connection associated to μ, and $\mathrm{Curv}^{\mathcal{A}} = d\mathcal{A} \neq 0$.

The compressed system

Identify $T^*S = TS$ via the induced metric μ_0. According to the general results on compression of internal symmetries:

The compressed Hamiltonian is

$$\mathcal{H}_{\mathrm{c}}(s,u) = \tfrac{1}{2}\langle \mathbb{I}u, u\rangle + \tfrac{1}{2}\langle \mathcal{A}_s(u), \mathcal{A}_s(u)\rangle_V$$

which is H-invariant. The compressed almost symplectic form is

$$\Omega_{\mathrm{nh}} = \Omega^S - \langle J_V \circ \mathrm{hl}^{\mathcal{A}}, \mathrm{Curv}_0^{\mathcal{A}}\rangle = \Omega^S + \langle \mathcal{A}, d\mathcal{A}\rangle_V$$

which is also H-invariant. The dynamics are given by X_{nh}:

$$i(X_{\mathrm{nh}})\Omega_{\mathrm{nh}} = d\mathcal{H}_{\mathrm{c}}.$$

Finally, there is a conserved quantity:

$$J_H : TS \to \mathfrak{h}^*$$

the standard momentum map. What about reduction?

Problem: $i(\zeta_Y)\Omega_{\mathrm{nh}} \neq d\langle J_H, Y\rangle$ whence $\Omega_{\mathrm{nh}}|J_H^{-1}(\lambda)$ is not horizontal for $J_H^{-1}(\lambda) \twoheadrightarrow J_H^{-1}(\lambda)/H_\lambda$.

Truncation

Let $\omega \in \Omega^1(S, \mathfrak{h})$ be the connection on $H \hookrightarrow S \twoheadrightarrow S^{n-1}$ associated to the biinvariant metric on S. Let $L = \omega : TS \to \mathfrak{h}$ viewed as a function, and $\mathrm{Curv}^\omega \in \Omega^2(S, \mathfrak{h})$ the curvature form.

Theorem

Let

$$\widetilde{\Omega} := \Omega^S - \langle L, \mathrm{Curv}^\omega \rangle.$$

Then the system $(TS, \widetilde{\Omega}, \mathcal{H}_{\mathrm{c}})$ satisfies:

(1) *$\widetilde{\Omega}$ is non-degenerate and H-invariant.*

(2) $i(X_{\mathrm{nh}})\widetilde{\Omega} = d\mathcal{H}_{\mathrm{c}}$.

(3) $i(\zeta_Y)\widetilde{\Omega} = d\langle J_H, Y \rangle$ *for all* $Y \in \mathfrak{h}$.

For (2) one uses an $\langle ., . \rangle$-ONB Y_α, Z_a that is adapted to the decomposition $\mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ and *right* extends this to a frame on S. Then

$$\mathcal{A}_s(u) = -\sum \langle \mathrm{Ad}(s)^{-1} Z_a, u \rangle e_a =: -\sum \eta^a_s(u) e_a.$$

Corollaries

Corollary

$(TS, \widetilde{\Omega}, \mathcal{H}_c)$ can be reduced to an almost Hamiltonian system on

$$J_H^{-1}(\mathcal{O})/H \cong TS^{n-1} \times_{S^{n-1}} (S \times_H \mathcal{O})$$

where the isomorphism depends on μ_0, and $\mathcal{O} \subset \mathfrak{h}^*$. In particular, $\mathcal{O} \hookrightarrow J_H^{-1}(\mathcal{O})/H \to TS^{n-1}$.

Can also do point reduction: $J_H^{-1}(\mathcal{O})/H = J_H^{-1}(\lambda)/H_\lambda$ for $\lambda \in \mathcal{O}$.

Corollary

When $\mathbb{I} = 1$, Chaplygin's ball problem is Hamiltonian after reduction by H.

Proof.

In this case $L = J_H$. ☐

In fact, this is not surprising: $X^{\mathcal{M}} = X_{\mathcal{H}}$. However, the space $\mathcal{D}$ does not have a symplectic interpretation, and $\mathcal{D}/V$ is *not* a symplectic quotient.

Hamiltonization for $n = 3$

Let $n = 3$, Consider the metric isomorphism $\check{\mu}_0 = \mathbb{I} + \mathcal{A}^*\mathcal{A} : TS \to T^*S \cong_{\langle\cdot,\cdot\rangle} TS$. Define

$$f(s) = (\det \check{\mu}_0(s))^{-\frac{1}{2}} \text{ for } s \in S$$

which is strictly positive, $H = S^1$-invariant, and drops to a function $S^2 \to \mathbb{R}$.

Theorem

Let $\lambda \in \mathfrak{h}^*$. Then $d(f\widetilde{\Omega})|J_H^{-1}(\lambda) = 0$. (Conormally closed)

This theorem is due to Borisov and Mamaev (2001, 2005), albeit in a different setting: Using Euler angles they showed that after an f-dependent *time reparametrization*, $d\tau = fdt$, the equations of motion for the Chaplygin ball system can be written with respect to a certain Poisson bracket on TS^2. It turns out that their Poisson bracket corresponds to the reduction of $f\widetilde{\Omega}$.