An (almost) symplectic view of Chaplygin’s ball

Based on:

Simon Hochgerner

EPFL, November 11, 2008
Contents

Part I: Reduction of $G$-Chaplygin systems

- $G$-Chaplygin systems
- Non-holonomic reduction: Compression
- $G$-Chaplygin systems with internal symmetries
- Reduction of internal symmetries: Truncation
Contents

Part I: Reduction of $G$-Chaplygin systems

* $G$-Chaplygin systems
* Non-holonomic reduction: Compression
* $G$-Chaplygin systems with internal symmetries
* Reduction of internal symmetries: Truncation

Part II: Chaplygin’s rolling ball

* The constraints
* The compressed system
* Reduction via truncation
**G-Chaplygin systems**

A *non-holonomic system* is a triple \((Q, L, \mathcal{D})\) where

- \(Q\) is an \(n\)-dimensional configuration manifold;
- \(\mathcal{D} \subset TQ\) is a smooth non-integrable distribution of constant rank;
- \(L = E_{\text{kin}} - E_{\text{pot}} : TQ \to \mathbb{R}\) is the Lagrangian, and we assume that \(E_{\text{kin}}\) defines a Riemannian metric \(\mu\) on \(Q\).

The equations of motion for a curve \(q(t)\) in \(Q\), such that \(q' \in \mathcal{D}\), are determined by the Lagrange-d’Alembert principle.

A *G-Chaplygin system* is a non-holonomic system \((Q, L, \mathcal{D})\) acted upon by a Lie group \(G\) in a free and proper fashion such that \(\mathcal{D}\) defines a connection on the principal bundle

\[
G \hookrightarrow Q \twoheadrightarrow Q/G =: S.
\]

(Also called the principal or purely kinematical case.) In particular, \(G\) acts by isometries on \((Q, \mu)\).

We do not require that \(\mathcal{D}\) is the mechanical connection.
The almost Hamiltonian formulation

Via the metric $\mu$ we can associate a Hamiltonian $\mathcal{H}$ to the Lagrangian,

$$\mathcal{H}(q, p) = \frac{1}{2}\mu_q(p, p) + V(q).$$

Let $\phi^a \in \Omega^1(Q)$, $a = 1, \ldots, \dim G$ be the components of the connection form such that $\mathcal{D} = \ker(\phi^a)_a$. In terms of local coordinates $(q^i, p_i)$ on $Q$ the equations of motion are given by

$$X^M(q, p) := \begin{pmatrix} q^i \\ p_i \end{pmatrix}' = \begin{pmatrix} -\frac{\partial H}{\partial p_i} - \sum_a \lambda_a \phi^a(\frac{\partial}{\partial q^i}) \end{pmatrix}$$

where the $\lambda_a$ are the Lagrange multipliers determined from the supplementary equations $\mu(\phi^a, p) = \phi^a(v) = 0$. Thus $X^M$ is tangent to

$$\mathcal{M} := \tilde{\mu}(\mathcal{D}).$$

Intrinsically this writes as

$$i(X^M)\Omega^Q = dH + \sum_a \lambda_a \tau^* \phi^a$$

where $\tau : T^*Q \to Q$. 
The Bates-Śniatycki construction

Let $\iota : M \hookrightarrow T^*Q$ and $A := (\phi^a)_a : TQ \to g$. The $G$-action induces an action on $M$ and there is an induced connection $\iota^*\tau^*A$ on the principal fiber bundle

$$G \hookrightarrow M \twoheadrightarrow M/G = T^*S.$$ 

Let

$$C := (\iota^*\tau^*A)^{-1}(0) \subset TM$$

be the horizontal space. Define $\Omega^C$ to be the fiber-wise restriction to $C$ of $\iota^*\Omega^Q$, and $(d\mathcal{H})^C$ the restriction of $d\mathcal{H}$ to $C$. Then

$$i(X^M)\Omega^C = (d\mathcal{H})^C.$$ 

Theorem ([BS93])

$\Omega^C : C \times C \to \mathbb{R}$ is non-degenerate.

Thus we can completely describe the dynamics in terms of the triple $(M, \Omega^C, \iota^*\mathcal{H})$ together with the above equation.

Can we reproduce this structure on $T^*S$?
Beside: The mechanical case

What happens when $\mathcal{A} : TQ \to \mathfrak{g}$ is the mechanical connection? Then $\mathcal{M} = J^{-1}(0)$ where $J : T^*Q \to \mathfrak{g}^*$ is the standard momentum map of the lifted $G$-action. Further, $\mathcal{C}$ is the associated horizontal space of $\mathcal{M} \to \mathcal{M}/G = T^*S$, and non-degeneracy of $\Omega^\mathcal{C}$ follows from the momentum map equation

$$i(\zeta_X)\Omega^Q = d\langle J, X \rangle$$

for all $X \in \mathfrak{g}$

which implies that $\text{Ver}(\mathcal{M} \to \mathcal{M}/G) = \ker i^*\Omega^Q$.

Therefore, when $\mathcal{A}$ is mechanical,

$$X^\mathcal{M} = X_\mathcal{H} = \hat{\Omega}^{-1}(d\mathcal{H}) \in \mathcal{C} \subset TT^*Q|_{\mathcal{M}}$$

since $X_\mathcal{H}$ is tangent to $\mathcal{M}$ by Noether’s Theorem.

When $\mathcal{A}$ is not the mechanical connection, $\mathcal{M} \neq J^{-1}(0)$, and the dynamics of $X^\mathcal{M}$ will, in general, differ from those of $X_\mathcal{H}$.
Compression

Let $\rho : \mathcal{M} \rightarrow \mathcal{M}/G = T^*S$ denote the projection. This map has a fiber-wise inverse:

$$hl^A : T^*S \cong_{\mu_0} TS \xrightarrow{hl^A} D \cong_{\mu} \mathcal{M}$$

where $\mu_0$ is the induced metric on $S$ and $hl^A$ is the horizontal lift.

Proposition ([BS93, K92])

- $\Omega^C$ descends to a non-degenerate two-form $\Omega_{nh}$ on $T^*S$.
- $\Omega_{nh} = \Omega^S - \langle J_G \circ hl^A, \text{Curv}_0^A \rangle$ where $\text{Curv}_0^A$ is the induced curvature form on $S$.
- $\iota^*\mathcal{H}$ drops to a function $\mathcal{H}_c$ on $T^*S$.
- The vector field $X^M$ is $\rho$-related to the vector field $X_{nh} := (\Omega_{nh})^{-1} d\mathcal{H}_c$.

The form $\Omega_{nh}$ is, in general, not closed. The non-holonomic system $(Q, L, D)$ is thus encoded in the almost Hamiltonian system $(T^*S, \Omega_{nh}, \mathcal{H}_c)$. 
How far is $\Omega_{nh}$ from closedness?

Let $\chi : T\mathcal{M} \to \mathcal{C}$ denote the projection onto the horizontal subspace. Then, alternatively to $\Omega^C$, one may also consider

$$i^*\Omega^Q \circ \Lambda^2\chi = -d(i^*\theta^Q) \circ \Lambda^2\chi = -d_A i^*\theta^Q$$

which is the covariant derivative of $i^*\theta^Q$. (This is the extension of $\Omega^C$ to $\Lambda^2 T\mathcal{M}$ by 0.)

Now, $\Omega_{nh}$ is closed iff $\rho^*\Omega_{nh}$ is closed. But,

$$d\rho^*\Omega_{nh} = d_A \rho^*\Omega_{nh} = d_A (i^*\Omega^Q \circ \Lambda^2\chi)$$

$$= -d_A^2 i^*\theta^Q.$$ 

Thus there are two scenarios which yield closedness of $\Omega_{nh}$:

- The connection $A$ is mechanical whence $i^*\Omega^Q$ is a horizontal form, that is, $i^*\Omega^Q \circ \Lambda^2\chi = i^*\Omega^Q$.

- The distribution $\mathcal{D}$ is integrable which means that the constraints are holonomic. For, $d^2_A = \chi^* \circ i(R) \circ d$ where $R = \zeta^G \circ \text{Curv}^A$ is the curvature.
Chaplygin systems with internal symmetries

We now furnish the $G$-Chaplygin system $(Q, L, D)$ with additional symmetries.

Suppose a Lie group $H$ acts properly and freely by two different actions, $l$ and $d$, on $Q$ such that both factor to one $H$-action on $S$ such that:

- $\pi : Q \rightarrow Q/G = S$ is $l$- and $d$-equivariant.

- $A : TQ \rightarrow g$ is $d$-equivariant with respect to a representation of $H$ on $g$. (This means that $D$ is $d$-invariant whence $d$ defines an external symmetry.)

- Additionally, $L$ is $l$- and $d$-invariant.

Conservation of internal symmetries

Since $l$ is internal it is true that $dJ_l X_M = 0$ where $J_l$ is the standard momentum map associated to the $l$-action on $T^*Q$. (Non-holonomic Noether Theorem)
Chaplygin systems with internal symmetries

We now furnish the $G$-Chaplygin system $(Q, L, \mathcal{D})$ with additional symmetries.

Suppose a Lie group $H$ acts properly and freely by two different actions, $l$ and $d$, on $Q$ such that both factor to one $H$-action on $S$ such that:

- $\pi : Q \to Q/G = S$ is $l$- and $d$-equivariant.
- $A : TQ \to g$ is $d$-equivariant with respect to a representation of $H$ on $g$. (This means that $\mathcal{D}$ is $d$-invariant whence $d$ defines an external symmetry.)
Chaplygin systems with internal symmetries

We now furnish the $G$-Chaplygin system $(Q, L, D)$ with additional symmetries.

Suppose a Lie group $H$ acts properly and freely by two different actions, $l$ and $d$, on $Q$ such that both factor to one $H$-action on $S$ such that:

- $\pi : Q \rightarrow Q/G = S$ is $l$- and $d$-equivariant.
- $\mathcal{A} : TQ \rightarrow g$ is $d$-equivariant with respect to a representation of $H$ on $g$. (This means that $D$ is $d$-invariant whence $d$ defines an external symmetry.)
- $\mathcal{A}.\zeta^l_Y = 0$ whence $l$ defines an internal symmetry.
Chaplygin systems with internal symmetries

We now furnish the $G$-Chaplygin system $(Q, L, \mathcal{D})$ with additional symmetries.

Suppose a Lie group $H$ acts properly and freely by two different actions, $l$ and $d$, on $Q$ such that both factor to one $H$-action on $S$ such that:

- $\pi : Q \to Q/G = S$ is $l$- and $d$-equivariant.
- $A : TQ \to g$ is $d$-equivariant with respect to a representation of $H$ on $g$. (This means that $\mathcal{D}$ is $d$-invariant whence $d$ defines an external symmetry.)
- $A.\zeta^l_Y = 0$ whence $l$ defines an internal symmetry.
- Additionally, $L$ is $l$- and $d$-invariant.

Conservation of internal symmetries

Since $l$ is internal it is true that $dJ^l_{\pi^*X} = 0$ where $J^l$ is the standard momentum map associated to the $l$-action on $T^*Q$.

(Non-holonomic Noether Theorem)
Chaplygin systems with internal symmetries

We now furnish the $G$-Chaplygin system $(Q, L, D)$ with additional symmetries.

Suppose a Lie group $H$ acts properly and freely by two different actions, $l$ and $d$, on $Q$ such that both factor to one $H$-action on $S$ such that:

- $\pi : Q \rightarrow Q/G = S$ is $l$- and $d$-equivariant.
- $A : TQ \rightarrow g$ is $d$-equivariant with respect to a representation of $H$ on $g$. (This means that $D$ is $d$-invariant whence $d$ defines an external symmetry.)
- $A.\zeta^l_Y = 0$ whence $l$ defines an internal symmetry.
- Additionally, $L$ is $l$- and $d$-invariant.

Conservation of internal symmetries

Since $l$ is internal it is true that $dJ^l.X^M = 0$ where $J^l$ is the standard momentum map associated to the $l$-action on $T^*Q$. (Non-holonomic Noether Theorem)
Compression of internal symmetries

How do these symmetries behave with respect to compression?

**Proposition**

- \( d \) induces an action on \( \mathcal{M} \) and \( \rho : \mathcal{M} \to T^* S \) is \( d \)-equivariant.
- \( \Omega_{\text{nh}} \) and \( \mathcal{H}_c \) are \( H \)-invariant. (Uses \( d \).)
- The standard momentum map \( J_H : T^* S \to \mathfrak{h}^* \) satisfies
  \[ J_H = (J_l|\mathcal{M}) \circ h l^A. \]
- Thus, \( dJ_H.\mathcal{X}_{\text{nh}} = 0. \)

\( l \) does not act on \( \mathcal{M} \) and \( J_d \) does not factor to \( J_H \).

Thus \( H \) is a symmetry group of the compressed system \((T^* S, \Omega_{\text{nh}}, \mathcal{H}_c)\) and \( J_H \) is a conserved quantity.

What about reduction by \( H \)? Can we reproduce this structure on \( J_H^{-1}(\lambda)/H_\lambda \) where \( \lambda \in \mathfrak{h}^* \)?
Reduction of internal symmetries?

Problem

\( J_H \) is not the momentum map of \( \Omega_{nh} \), i.e.,

\[
i(\zeta_Y)\Omega_{nh} \neq d\langle J_H, Y \rangle
\]

for general \( Y \in \mathfrak{h} \). That is, \( \Omega_{nh}|_{J_H^{-1}(\lambda)} \) is not horizontal with respect to \( J_H^{-1}(\lambda) \to J_H^{-1}(\lambda)/H_\lambda \).

Solution

Replace \( \Omega_{nh} \) by \( \tilde{\Omega} \) such that:

1. \( \tilde{\Omega} \) is non-degenerate.
2. \( i(X_{nh})\tilde{\Omega} = d\mathcal{H}_c \).
3. \( \tilde{\Omega} \) is \( H \)-invariant.
4. \( i(\zeta_Y)\tilde{\Omega} = d\langle J_H, Y \rangle \) for all \( Y \in \mathfrak{h} \).

Does such an \( \tilde{\Omega} \) exist?
Reduction via truncation

Theorem

Suppose there is a connection $\sigma \in \Omega^1(T^*S, \mathfrak{h})$ on the principal bundle $T^*S \to (T^*S)/H$ such that $X_{nh}$ is horizontal. Then the truncated form

$$\tilde{\Omega} := \Omega^S - \langle J_G \circ hl^A, \text{Curv}_0^A \rangle \circ \Lambda^2 \chi$$

satisfies (1)-(4); $\chi$ is the horizontal projection associated to $\sigma$.

Proof.

(1), (3), (4) are immediate.

For (2) we need to show that $\Omega_{nh}(X_{nh}, \xi) = \tilde{\Omega}(X_{nh}, \xi)$ for all $\xi$.

When $\xi$ is horizontal this is obvious. Assume that $\xi = \zeta_Y$. Then:

$$0 = dH_c \cdot \zeta_Y = \Omega_{nh}(X_{nh}, \zeta_Y)$$

$$0 = -\langle dJ_H \cdot X_{nh}, Y \rangle = \Omega^S(X_{nh}, \zeta_Y) = \tilde{\Omega}(X_{nh}, \zeta_Y).$$

In particular, $\langle J_G \circ hl^A, \text{Curv}_0^A \rangle(X_{nh}, \zeta_Y) = 0$. 
Reduction via truncation: Existence

Let $\mathcal{E} := X_{\text{nh}}^{-1}(\text{Ver}(H))$, and $\mathcal{U} := (T^* S) \setminus \mathcal{E}$ which is open.

Theorem

- $\mathcal{U}$ and $\mathcal{E}$ are both $H$- and $X_{\text{nh}}$-invariant.
- On the principal bundle $\mathcal{U} \rightarrow \mathcal{U}/H$ there is a connection $\sigma$ such that $X_{\text{nh}}$ is horizontal.

Conclusion

We may replace $(T^* S, \Omega_{\text{nh}}, \mathcal{H}_c)$ by treating two separate problems: Firstly, the problem on $\mathcal{E}$ which are the relative equilibria; secondly we can do almost Hamiltonian reduction with respect to the $H$-action of $(\mathcal{U}, \widetilde{\Omega}_\mathcal{U}, \mathcal{H}_c|\mathcal{U})$.

This shows that the form $\Omega_{\text{nh}}$ is ‘not optimal’ for the description of Chaplygin systems with internal symmetries. It carries ‘too much information’.
Part II: Chaplygin’s rolling ball

The Chaplygin ball is a round ball whose center of mass lies at the geometric center and which rolls without slipping or sliding on a horizontal table.

Constraints

The horizontal component of the contact point’s angular velocity equals minus the ball’s linear velocity. Or: The contact point has zero total velocity.
The $n$-dimensional Chaplygin ball

The configuration space of the $n$-dimensional Chaplygin ball is

\[ Q := S \times V := \text{SO}(n) \times \mathbb{R}^{n-1}. \]

A curve \((s(t), x(t))\) is an allowed motion iff

\[ (s's^{-1}, x') \in \tilde{D} := \{ (\tilde{u}, x') \in so(n)_R \times V : \tilde{u}.e_n = (x', 0)^t \}. \]

Here \(so(n)_R\) is the Lie algebra of right invariant vector fields on \(S\). In terms of the left trivialization, \(TS = S \times so(n)\), the Hamiltonian reads

\[ H = \frac{1}{2}\langle IIu, u \rangle + \frac{1}{2}\langle x', x' \rangle_V \]

where \(u = s^{-1}s'\), \(\langle ., . \rangle\) is minus the Killing form on \(so(n)\) and \(II\) is the inertia matrix. Thus the metric is

\[ \mu = \langle II ., . \rangle + \langle ., . \rangle_V. \]

(Units such that mass and radius equal to 1.)
Left vs. Right

Let $\tilde{A} : S \times \mathfrak{so}(n)_R \to V, (s, \tilde{u}) \mapsto -\text{pr}_V(\tilde{u}.e_n)$. Then

$$\tilde{D} = \{(s, \tilde{u}, x, -\tilde{A}(\tilde{u}))\}.$$ 

In the left trivialization this becomes

$$A : TS = S \times \mathfrak{so}(n) \longrightarrow S \times \mathfrak{so}(n)_R \longrightarrow V$$

$$(s, u) \longmapsto (s, \text{Ad}(s)u) \longmapsto -\text{pr}_V((\text{Ad}(s)u).e_n),$$

and $\tilde{D}$ translates to

$$D = \{(s, u, x, -A_s(u))\} \subset TS \times TV = TQ$$

which is a (right invariant) non-integrable distribution on $Q$.

The Chaplygin ball is the non-holonomic system described by the triple $(Q, D, \frac{1}{2}\| \cdot \|_\mu^2)$. What about symmetries?
Symmetries

\((Q, \mathcal{D}, \frac{1}{2}|| \cdot \||^2_\mu)\) is a \(G\)-Chaplygin system with \(G = V\).

1. \(\mathcal{A} : TS \rightarrow V\) is a connection form on the principal fiber bundle \(V \hookrightarrow Q \rightarrow S\).
2. \(\mathcal{A}\) is right invariant.

The group \(H = \{h \in S : h.e_n = e_n\} = SO(n - 1)\) acts through two different actions on \(Q\):

3. The \(l\)-action: \(l_h(s, x) = (hs, x)\). This action generates internal symmetries: \(\mathcal{A}_Y = \tilde{\mathcal{A}}Y = 0\) for all \(Y \in \mathfrak{h}\).
4. The \(d\)-action: \(d_h(s, x) = (hs, hx)\). This action generates external symmetries. \(\mathcal{A}(hs, u) = h.\mathcal{A}(s, u)\) for all \(h \in H\). Thus \(\mathcal{D}\) is invariant under the \(d\)-action.

Notice that \(\mathcal{A}\) is never the mechanical connection associated to \(\mu\), and \(\text{Curv}^\mathcal{A} = d\mathcal{A} \neq 0\).
The compressed system

Identify $T^*S = TS$ via the induced metric $\mu_0$. According to the general results on compression of internal symmetries:

The compressed Hamiltonian is

$$\mathcal{H}_c(s, u) = \frac{1}{2}\langle \Pi u, u \rangle + \frac{1}{2}\langle A_s(u), A_s(u) \rangle_V$$

which is $H$-invariant. The compressed almost symplectic form is

$$\Omega_{nh} = \Omega^S - \langle J_V \circ h l^A, \text{Curv}_0^A \rangle = \Omega^S + \langle A, dA \rangle_V$$

which is also $H$-invariant. The dynamics are given by $X_{nh}$:

$$i(X_{nh})\Omega_{nh} = d\mathcal{H}_c.$$ 

Finally, there is a conserved quantity:

$$J_H : TS \to \mathfrak{h}^*$$

the standard momentum map. What about reduction?

Problem: $i(\zeta_Y)\Omega_{nh} \neq d\langle J_H, Y \rangle$ whence $\Omega_{nh}|J_H^{-1}(\lambda)$ is not horizontal for $J_H^{-1}(\lambda) \to J_H^{-1}(\lambda)/H\lambda$. 
Truncation

Let \( \omega \in \Omega^1(S, \mathfrak{h}) \) be the connection on \( H \hookrightarrow S \twoheadrightarrow S^{n-1} \) associated to the biinvariant metric on \( S \). Let \( L = \omega : TS \to \mathfrak{h} \) viewed as a function, and \( \text{Curv}^\omega \in \Omega^2(S, \mathfrak{h}) \) the curvature form.

**Theorem**

Let

\[
\tilde{\Omega} := \Omega^S - \langle L, \text{Curv}^\omega \rangle.
\]

Then the system \((TS, \tilde{\Omega}, \mathcal{H}_c)\) satisfies:

1. \( \tilde{\Omega} \) is non-degenerate and \( H \)-invariant.
2. \( i(X_{nh})\tilde{\Omega} = d\mathcal{H}_c \).
3. \( i(\zeta_Y)\tilde{\Omega} = d\langle J_H, Y \rangle \) for all \( Y \in \mathfrak{h} \).

For (2) one uses an \( \langle ., . \rangle \)-ONB \( Y_\alpha, Z_a \) that is adapted to the decomposition \( \mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp \) and right extends this to a frame on \( S \). Then

\[
A_s(u) = - \sum \langle \text{Ad}(s)^{-1}Z_a, u \rangle e_a =: - \sum \eta_s^a(u)e_a.
\]
Corollaries

Corollary

\((TS, \tilde{\Omega}, \mathcal{H}_c)\) can be reduced to an almost Hamiltonian system on

\[ J_H^{-1}(\mathcal{O})/H \cong TS^{n-1} \times S^{n-1} (S \times H \mathcal{O}) \]

where the isomorphism depends on \(\mu_0\), and \(\mathcal{O} \subset \mathfrak{h}^*\). In particular, \(\mathcal{O} \hookrightarrow J_H^{-1}(\mathcal{O})/H \rightarrow TS^{n-1}\).

Can also do point reduction: \(J_H^{-1}(\mathcal{O})/H = J_H^{-1}(\lambda)/H\lambda\) for \(\lambda \in \mathcal{O}\).

Corollary

When \(\mathbb{I} = 1\), Chaplygin’s ball problem is Hamiltonian after reduction by \(H\).

Proof.

In this case \(L = J_H\). \(\square\)

In fact, this is not surprising: \(X^M = X_H\). However, the space \(\mathcal{D}\) does not have a symplectic interpretation, and \(\mathcal{D}/\mathcal{V}\) is not a symplectic quotient.
Hamiltonization for $n = 3$

Let $n = 3$, Consider the metric isomorphism
\[ \mu_0 = \mathbb{I} + \mathcal{A}^* \mathcal{A} : TS \to T^* S \cong \langle \cdot, \cdot \rangle \ TS. \]

Define
\[ f(s) = (\det \mu_0(s))^{-\frac{1}{2}} \text{ for } s \in S \]

which is strictly positive, $H = S^1$-invariant, and drops to a function $S^2 \to \mathbb{R}$.

Theorem

Let $\lambda \in \mathfrak{h}^*$. Then $d(f\widetilde{\Omega})|_{J_H^{-1}(\lambda)} = 0$. (Conormally closed)

This theorem is due to Borisov and Mamaev (2001, 2005), albeit in a different setting: Using Euler angles they showed that after an $f$-dependent time reparametrization, $d\tau = fd\tau$, the equations of motion for the Chaplygin ball system can be written with respect to a certain Poisson bracket on $TS^2$. It turns out that their Poisson bracket corresponds to the reduction of $f\widetilde{\Omega}$. 