

**SINGULAR COTANGENT BUNDLE REDUCTION
&
SPIN CALOGERO-MOSER SYSTEMS**

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ABSTRACT. We develop a bundle picture for singular symplectic quotients of cotangent bundles acted upon by cotangent lifted actions for the case that the configuration manifold is of single orbit type. Furthermore, we give a formula for the reduced symplectic form in this setting. As an application of this bundle picture we consider Calogero-Moser systems with spin associated to polar representations of compact Lie groups.

1. INTRODUCTION

Let G be a Lie group acting properly on a configuration manifold Q . Consider the cotangent lifted G -action on T^*Q . This action is Hamiltonian with respect to the standard exact symplectic form on T^*Q and with equivariant momentum map denoted by $\mu : T^*Q \rightarrow \mathfrak{g}^*$. Assume \mathcal{O} is a coadjoint orbit contained in the image of μ . The first part of this paper is concerned with the study of the singular symplectic quotient

$$\mu^{-1}(\mathcal{O})/G =: T^*Q//_{\mathcal{O}}G.$$

Indeed, this quotient cannot be a smooth manifold, in general, since we do not assume the G -action on Q to be free. However, we can apply the theory of singular symplectic reduction as developed by Sjamaar and Lerman [41], Bates and Lerman [6], and Ortega and Ratiu [30] (see also Theorem 3.1), and this exhibits $T^*Q//_{\mathcal{O}}G$ to be a Whitney stratified space with strata of the form

$$(\mu^{-1}(\mathcal{O}) \cap (T^*Q)_{(L)})/G =: (T^*Q//_{\mathcal{O}}G)_{(L)}$$

where (L) is an element of the isotropy lattice of the G -action on T^*Q .

One of the aims of this paper is to develop a bundle picture for the reduced phase space $T^*Q//_{\mathcal{O}}G$, i.e., to obtain a fiber bundle $T^*Q//_{\mathcal{O}}G \rightarrow T^*(Q/G)$ in a suitable (singular) sense. One can hope to construct such a fiber bundle in the presence of an additionally chosen generalized connection form (see Section 5) on $Q \rightarrow Q/G$. However, for reasons explained in Remark 4.5 there cannot exist a surjection $T^*Q//_{\mathcal{O}}G \rightarrow T^*(Q/G)$ with locally constant fiber type for general proper G -actions. In Remark 4.5(F) we state a weak substitute of a bundle picture for $T^*Q//_{\mathcal{O}}G$.

To obtain a bundle picture and a useful description of the reduced phase space $T^*Q//_{\mathcal{O}}G$ we have to assume that the base manifold Q is of single orbit type, that is, $Q = Q_{(H)}$ for a subgroup H of G . Assuming this we get a first result that says that, locally,

$$\mathcal{O}//_0H \longrightarrow T^*Q//_{\mathcal{O}}G \longrightarrow T^*(Q/G)$$

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is a symplectic fiber bundle (Theorem 4.4). This result is obtained by applying the Palais Slice Theorem to the G -action on the base space Q , and then using the Singular Commuting Reduction Theorem of Section 3. This is an inroad that was also taken by Schmah [39] to get a local description of $T^*Q//_{\mathcal{O}}G$.

However, one can also give a global symplectic description of the reduced phase space and this is done in Section 5. This follows an approach that is generally called Weinstein construction ([47]). (We do not consider the closely related construction of Sternberg [43].) In the case that the G -action on the configuration space Q is free this global description was first given by Marsden and Perlmutter [23]. Their result says that the choice of a principal bundle connection on $Q \rightarrow Q/G$ yields a realization of the symplectic quotient $T^*Q//_{\mathcal{O}}G$ as a fibered product

$$T^*(Q/G) \times_{Q/G} (Q \times_G \mathcal{O}),$$

and they compute the reduced symplectic structure in terms of data intrinsic to this realization – [23, Theorem 4.3].

In the presence of a single non-trivial isotropy (H) on the configuration space one obtains a non-trivial isotropy lattice on T^*Q whence the symplectic reduction of T^*Q is to be carried out in the singular context of [41, 6, 30]. The result is the following: The choice of a generalized connection form (see Section 5) on $Q \rightarrow Q/G$ yields a realization of each symplectic stratum $(T^*Q//_{\mathcal{O}}G)_{(L)}$ of the reduced space as a fibered product

$$(W//_{\mathcal{O}}G)_{(L)} = T^*(Q/G) \times_{Q/G} (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{g}_q)_{(L)}/G$$

where

$$W := (Q \times_{Q/G} T^*(Q/G)) \times_Q \bigsqcup_{q \in Q} \text{Ann } \mathfrak{g}_q \cong T^*Q$$

as symplectic manifolds with a Hamiltonian G -action. Moreover, we compute the reduced symplectic structure in terms intrinsic to this realization. This is the content of Theorem 5.6. Even though we have to restrict to single orbit type manifolds this result is quite general in the sense that it is valid for arbitrary coadjoint orbits \mathcal{O} .

The first to have studied symplectic reduction of cotangent bundles for non-free actions seems to have been Montgomery [27]. Using the point reduction approach this paper gives conditions under which the reduced phase space carries a smooth manifold structure. The first to study this subject in the context of singular symplectic reduction as developed by [41, 6, 30] is Schmah [39] who proves a cotangent bundle specific slice theorem at points whose momentum values are fully isotropic. The other important paper on singular cotangent bundle reduction is by Perlmutter, Rodriguez-Olmos and Sousa-Diaz [35]. By restricting to do reduction at fully isotropic values of the momentum map $\mu : T^*Q \rightarrow \mathfrak{g}^*$ they are able to drop all assumptions on the isotropy lattice of the G -action on Q , and give a very complete description of the reduced symplectic space.

As an application of the bundle picture found in Theorem 5.6 we consider Calogero-Moser systems with spin in Section 6. In fact, it was an idea of Alekseevsky, Kriegl, Losik, Michor [2] to consider polar representations of compact Lie groups G on a Euclidean vector space V to obtain new versions of Calogero-Moser models. We make these ideas precise by using the singular cotangent bundle reduction machinery. Thus let Σ be a section for the G -action in V , let C be a Weyl chamber in this section, and put $M := Z_G(\Sigma)$. Under a strong but not impossible condition on a chosen coadjoint orbit in \mathfrak{g}^* we get

$$T^*V//_{\mathcal{O}}G = T^*C_r \times \mathcal{O}//_0M$$

from the general theory (Theorem 5.6), where C_r denotes the sub-manifold of regular elements in C . This is the effective phase space of the spin Calogero-Moser system. The corresponding Calogero-Moser function is obtained as a reduced Hamiltonian from the free Hamiltonian on T^*V . The resulting formula is

$$H_{\text{CM}}(q, p, [Z]) = \frac{1}{2} \sum_{i=1}^l p_i^2 + \frac{1}{2} \sum_{\lambda \in R} \frac{\sum_{i=1}^{k_\lambda} z_\lambda^i z_\lambda^i}{\lambda(q)^2}.$$

This is made precise with the necessary notation in Section 6.

It was first observed by Kazhdan, Kostant and Sternberg [17] that one can obtain Calogero-Moser models via Hamiltonian reduction of $T^*\mathfrak{g}$ or T^*G . (See also Subsections 6.A and 6.B.) In fact, [17] chose their example such that the reduction procedure yields a smooth symplectic manifold, and moreover there appear no spin variables, i.e., $\mathcal{O} //_0 H = \{\text{point}\}$. This reduction approach was further pursued in [2] as alluded to above. However, these spin models are Hamiltonian systems on non-smooth spaces, and it was not clear in which sense one should regard the reduced systems as being Hamiltonian systems. Moreover, the precise form of the singularities was not clear. These questions are answered by Theorem 5.6: The reduced system is a stratified Hamiltonian system, the strata of the reduced phase space are described, and when the system is restricted to a stratum it is a Hamiltonian system in the usual sense on this stratum. Moreover, the stratification is Whitney whence by [41] the singularities which appear are of conic form.

As an interesting side product (Remark 6.1) our (singular) cotangent bundle reduction approach yields a connection to the r -matrix theoretic construction of Calogero-Moser models of Li and Xu [19, 20]. This connection is new and interesting since it explains in geometric terms why solutions of the classical dynamical Yang-Baxter equation lead to Calogero-Moser systems.

Finally, we use a result on non-commutative integrability from Zung [50, Theorem 2.3] to show that these Calogero-Moser systems are integrable in the non-commutative sense.

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2. PRELIMINARIES AND NOTATION

All manifolds to be considered are Hausdorff, para-compact, finite dimensional, and smooth in the C^∞ -sense. We do not assume that manifolds are connected but allow for finitely many connected components of varying (finite) dimension. Thus the dimension of a manifold is only locally constant. A proper G -space M is a manifold M acted upon properly by a Lie group G . For proper G -spaces the Slice Theorem ([31, 32, 11]) holds, and we shall make frequent use of this theorem.

Let M be a proper G -space. An ISOTROPY CLASS (H) of the G -action on M is a conjugacy class of an isotropy subgroup G_x of a point $x \in M$, that is, $(H) = \{gG_xg^{-1} : g \in G\}$. If we want to explicit that (H) is the conjugacy class of H with respect to G we shall write $(H)^G$. The ISOTROPY LATTICE $\mathcal{IL}(M)$ is defined to be the lattice consisting of all isotropy classes (H) of the G -space M . For $(H) \in \mathcal{IL}(M)$

the ORBIT TYPE sub-manifold is

$$(1) \quad M_{(H)} := M_{(H)G} := \{x \in M : G_x \text{ is conjugate to } H \text{ within } G\},$$

the SYMMETRY TYPE sub-manifold is $M_H := \{x \in M : G_x = H\}$, and the FIXED POINT sub-manifold is $M^H := \{x \in M : H \subset G_x\}$. More generally, let N be a closed topological subspace of M and let K be a compact subgroup of G which acts (continuously) on N . We may view M also as a proper K -space, and thus obtain an orbit type stratification of M with respect to the K -action. Since N is K -invariant this induces a decomposition of N according to orbit types. Let $N_{(L_0)} = N_{(L_0)K} = N \cap M_{(L_0)K}$ be such an orbit type stratum, that is $L_0 = K_x$ for some $x \in N$. With regard to $L_0 \subset K$ we will be concerned with the GENERALIZED ISOTROPY CLASS

$$(2) \quad (L_0)_K^G := \{L \subset K : \text{there is } g \in G \text{ s.t. } gL_0g^{-1} = L\}$$

and the corresponding generalized orbit type space

$$(3) \quad N_{(L_0)_K^G} := \{z \in N : K_z \in (L_0)_K^G\}.$$

Clearly, $N_{(L_0)_K^G}$ itself decomposes into orbit type strata with respect to the induced K -action.

Using the Slice Theorem one can show (see [11]) that the stratification of M into orbit type sub-manifolds forms a Whitney stratification. Likewise the stratification of the topological space M/G into strata of the type $M_{(H)}/G$ where $(H) \in \mathcal{IL}(M)$ forms a Whitney stratification. Thus M/G becomes a stratified space, i.e., a stratified space such that the stratification satisfies the Whitney conditions. See [24, 15, 36, 37] for more on stratified spaces. If X and Y are stratified spaces a STRATIFIED MAP $\phi : X \rightarrow Y$ is a continuous map which respects the stratifications, that is, the pre-image under ϕ of every stratum of Y decomposes into a union of strata of X . For example, the orbit projection map $\pi : M \rightarrow M/G$ is a stratified map.

If the action is written as $l : G \times M \rightarrow M$, $(k, x) \mapsto l(k, x) = l_k(x) = l^x(k) = k.x$ we can tangent bundle lift it via $k.(x, v) := (k.x, k.v) := Tl_k.(x, v) = (l_k(x), T_x l_k.v)$ for $(x, v) \in TM$ to an action on TM . As the action consists of transformations by diffeomorphisms it may also be lifted to the cotangent bundle. This is the cotangent lifted action which is defined by $k.(x, p) := (k.x, k.p) := T^*l_k.(x, p) = (k.x, T_{k.x}^* l_{k^{-1}}.p)$ where $(x, p) \in T^*M$. Our notation for the fundamental vector field is $\zeta_X(x) := \zeta(x)(X) := \frac{\partial}{\partial t} |_{0} l(\exp(+tX), x) = T_e l^x(X)$ where $X \in \mathfrak{g}$.

3. SINGULAR SYMPLECTIC REDUCTION

Let (M, ω) be a connected symplectic manifold, and G a Lie group that acts on (M, ω) in a proper and Hamiltonian fashion such that there is an equivariant momentum map $J : M \rightarrow \mathfrak{g}^*$.

The very strong machinery of singular symplectic reduction is (for the case of compact G) due to Sjamaar and Lerman [41] who prove that the singular symplectic quotient is a Whitney stratified space that has symplectic manifolds as its strata. This result which is the Singular Reduction Theorem was then generalized to the case of proper actions by Bates and Lerman [6], Ortega and Ratiu [30], and others.

Theorem 3.1 (Singular symplectic reduction). *Let (H) be in the isotropy lattice of the G -action on M , and suppose that $J^{-1}(\mathcal{O}) \cap M_{(H)} \neq \emptyset$ for a coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$. Then the following are true.*

- The subset $J^{-1}(\mathcal{O}) \cap M_{(H)}$ is an initial sub-manifold of M .

- The topological quotient $(J^{-1}(\mathcal{O}) \cap M_{(H)})/G$ has a unique smooth structure such that the projection map

$$J^{-1}(\mathcal{O}) \cap M_{(H)} \xrightarrow{\pi} (J^{-1}(\mathcal{O}) \cap M_{(H)})/G$$

is a smooth surjective submersion.

- Let $\iota : J^{-1}(\mathcal{O}) \cap M_{(H)} \hookrightarrow M$ denote the inclusion mapping. Then $(J^{-1}(\mathcal{O}) \cap M_{(H)})/G$ carries a symplectic structure ω_0 which is uniquely characterized by the formula

$$\pi^* \omega_0 = \iota^* \omega - (J|(J^{-1}(\mathcal{O}) \cap M_{(H)}))^* \Omega^{\mathcal{O}}$$

where $\Omega^{\mathcal{O}}$ is the canonical (positive Kirillov-Kostant-Souriau) symplectic form on \mathcal{O} .

- Consider a G -invariant function $H \in C^\infty(M)^G$. Then the flow of the Hamiltonian vector field ∇_H^ω leaves the connected components of $J^{-1}(\mathcal{O}) \cap M_{(H)}$ invariant. Moreover, H factors to a smooth function h on the quotient $(J^{-1}(\mathcal{O}) \cap M_{(H)})/G$. Finally, ∇_H^ω and the Hamiltonian vector field to h are related via the canonical projection π , whence the flow of the former projects to the flow of the latter.
- The collection of all strata of the form $(J^{-1}(\mathcal{O}) \cap M_{(H)})/G$ constitutes a Whitney stratification of the topological space $J^{-1}(\mathcal{O})/G$.

Proof. This theorem is contained in [30, Section 8]. See also [6, Corollary 14] and [41]. \square

In fact, [30] state the above theorem only for connected components of strata. This is so because they do not require the momentum map J to be equivariant with respect to the co-coadjoint action on \mathfrak{g}^* .

As a matter of convention we write shorthand $M//_{\mathcal{O}}G := J^{-1}(\mathcal{O})/G$ for the reduced space of M with respect to the Hamiltonian action by G . If \mathcal{O} is the coadjoint orbit passing through α then we shall also abbreviate $J^{-1}(\alpha)/G_\alpha = M//_\alpha G = M//_{\mathcal{O}}G$.

Theorem 3.2 (Singular commuting reduction). *Let G and H be Lie groups that act properly and by symplectomorphisms on (M, ω) with momentum maps J_G and J_H respectively. Assume that the actions commute, that J_G is H -invariant, and that J_H is G -invariant. Let $\alpha \in \mathfrak{g}^*$ be in the image of J_G and $\beta \in \mathfrak{h}^*$ in the image of J_H .*

Then the G action drops to a Poisson action on $M//_\beta H$ and J_G factors to a momentum map j_G for the induced action. Likewise, the H action drops to a Poisson action on $M//_\alpha G$ and J_H factors to a momentum map j_H for the induced action. Furthermore, we have

$$(M//_\alpha G)//_\beta H \cong M//_{(\alpha, \beta)}(G \times H) \cong (M//_\beta H)//_\alpha G$$

as symplectic stratified spaces.

Proof. An outline of a proof of this result is given in [41, Section 4] for the case that G and H are compact. Using the machinery of singular symplectic reduction for proper Hamiltonian actions as described in [30] the proof of [41] extends to the more general setting. \square

4. THE BUNDLE PICTURE

From now on let G be a Lie group acting properly from the left on a manifold Q . The G action then induces a Hamiltonian action on the cotangent bundle T^*Q by cotangent lifts. This means that the lifted action respects the canonical symplectic

form $\Omega = -d\theta$ on T^*Q where θ is the Liouville form on T^*Q , and, moreover, there is an equivariant momentum map $\mu : T^*Q \rightarrow \mathfrak{g}^*$ given by $\langle \mu(q, p), X \rangle = \theta(\zeta_X^{T^*Q})(q, p) = \langle p, \zeta_X(q) \rangle$ where $(q, p) \in T^*Q$, $X \in \mathfrak{g}$, ζ_X is the fundamental vector field associated to the G -action on Q , and $\zeta_X^{T^*Q} \in \mathfrak{X}(T^*Q)$ is the fundamental vector field associated to the cotangent lifted action.

In this section we want to apply the Slice Theorem ([32, 11, 30]) to the action of G on Q to get a local model of the singular symplectic reduced space $T^*Q//_{\mathcal{O}}G = \mu^{-1}(\mathcal{O})/G$ where \mathcal{O} is a coadjoint orbit in the image of μ .

Thus we consider a tube U in Q around an orbit $G.q$ with $G_q = H$. And we denote the slice at q by S such that

$$U \cong G \times_H S$$

as G -spaces. Here the action on U is given by the restriction of the G -action on Q to the invariant neighborhood U of $G.q$. On the other hand, the action on $G \times_H S$ is given by $g.(k, s)_H = [(gk, s)]_H$ where $g \in G$ and $[(k, s)]_H$ denotes the class of $(k, s) \in G \times S$ in $G \times_H S$. Moreover, note that the H -action on $G \times S$ is given by $h.(k, s) = (kh^{-1}, h.s)$ where $h \in H$. (See [32, 11, 30].) In particular, it follows that $U/G \cong S/H$ as stratified spaces with smooth structure. (See [36, 37].)

Assume for a moment that the action by G on U is free, whence $U \cong G \times S$. Let $\mu : T^*U \rightarrow \mathfrak{g}^*$ be the canonical momentum mapping, $\lambda \in \mathfrak{g}^*$ a regular value in the image of μ , and \mathcal{O} the coadjoint orbit passing through λ . Then we have

$$\begin{aligned} (T^*U)//_{\mathcal{O}}G &= (T^*U)//_{\lambda}G = (T^*G \times T^*S)//_{\lambda}G = (T^*G)//_{\lambda}G \times T^*S \\ &= \mathcal{O} \times T^*(U/G) \end{aligned}$$

as symplectic spaces; since $T^*G//_{\lambda}G = \mathcal{O}$. The aim of this section is to drop the freeness assumption. To do so we will take the same approach as Schmah [39] and use singular commuting reduction.

Now we return to the case where $U = G \times_H S$ as introduced above. On $G \times S$ we will be concerned with two commuting actions. These are

$$(4) \quad \lambda : G \times G \times S \longrightarrow G \times S, \quad \lambda_g(k, s) = (gk, s)$$

$$(5) \quad \tau : H \times G \times S \longrightarrow G \times S, \quad \tau_h(k, s) = (kh^{-1}, h.s).$$

These actions obviously commute. The latter, i.e., τ is called the twisted action by H on $G \times S$. We can cotangent lift λ and τ to give Hamiltonian transformations on $T^*(G \times S)$ with momentum mappings J^λ and J^τ , respectively. By left translation we trivialize $T^*(G \times S) = (G \times \mathfrak{g}^*) \times T^*S$.

To facilitate the notation we will denote the cotangent lifted action of λ, τ again by λ, τ respectively.

Lemma 4.1. *Let $(k, \eta; s, p) \in G \times \mathfrak{g}^* \times T^*S$. Then we have the following formulas.*

$$(6) \quad J^\lambda(k, \eta; s, p) = \text{Ad}(k^{-1})^*.\eta =: \text{Ad}^*(k).\eta \in \mathfrak{g}^*,$$

$$(7) \quad J^\tau(k, \eta; s, p) = -\eta|\mathfrak{h} + \mu(s, p) \in \mathfrak{h}^*$$

where μ is the canonical momentum map on T^*S . Moreover, the actions λ and τ commute, and J^λ is H -invariant and J^τ is G -invariant.

Since the formula of the canonical momentum map on T^*S with regard to the H -action is the same as that on T^*U with regard to the G -action we use the same symbol μ for both these maps. It will be clear from the context whether μ denotes the H - or the G -momentum map whence this will not cause any confusion.

Proof. We denote the left action by G on itself by L , the right action by R , and the conjugate action by conj . In this notation we then have $\text{Ad}(k).X = T_e \text{conj}_k.X$, and

$\text{conj}_k = L_k \circ R^{k^{-1}} = R^{k^{-1}} \circ L_k$. It is straightforward to verify that the cotangent lifted actions of L and R on $T^*G = G \times \mathfrak{g}^*$ are given by

$$\begin{aligned} T^*L_g(k, \eta) &= (gk, \eta) = (gk, -\eta \circ \zeta^{R^{-1}}(g)) \\ T^*R^g(k, \eta) &= (kg, \text{Ad}^*(g^{-1}).\eta) = (kg, \eta \circ \zeta^L(g^{-1})) \end{aligned}$$

where ζ^L and ζ^R denote the fundamental vector field mappings associated to L and R respectively. Using the left trivialization $TG = G \times \mathfrak{g}$ we thus find that

$$\begin{aligned} \langle J^\lambda(k, \eta; s, p), X \rangle &= \langle \eta, \zeta_X^L(k) \rangle = \langle \eta, T_k L_{k^{-1}} \frac{\partial}{\partial t} \Big|_0 \exp(tX)k \rangle \\ &= \langle \eta, T_e(L_{k^{-1}} \circ R^k).X \rangle = \langle \text{Ad}^*(k).\eta, X \rangle \end{aligned}$$

for all $X \in \mathfrak{g}$ which shows the first claim. Likewise, it furthermore follows that $\langle J^\tau(k, \eta; s, p), Z \rangle = \langle -\eta, Z \rangle + \langle p, \zeta_X(s) \rangle$ for all $Z \in \mathfrak{h}$. The invariance of J^λ and J^τ is immediate from the formulas of the trivialized cotangent lifted actions. \square

Corollary 4.2. *Let $\alpha \in \mathfrak{g}^*$ and $\beta \in \mathfrak{h}^*$ such that α, β is in the image of J^λ, J^τ respectively. Then the following are true.*

- (1) *The action λ descends to a Hamiltonian action on the Marsden-Weinstein reduced space $T^*(G \times S) //_{\beta} H$. Moreover, J^λ factors to a momentum map $j_\lambda : T^*(G \times S) //_{\beta} H \rightarrow \mathfrak{g}^*$ for this action.*
- (2) *The action τ descends to a Hamiltonian action on the Marsden-Weinstein reduced space $T^*(G \times S) //_{\alpha} G$. Moreover, J^τ factors to a momentum map $j_\tau : T^*(G \times S) //_{\alpha} G \rightarrow \mathfrak{h}^*$ for this action.*
- (3) *The product action $G \times H \times T^*(G \times S) \rightarrow T^*(G \times S)$, $(k, h, u) \mapsto \lambda_k.\tau_h.u$ is Hamiltonian with momentum map (J^λ, J^τ) . Moreover,*

$$(T^*(G \times S) //_{\alpha} G) //_{\beta} H = T^*(G \times S) //_{(\alpha, \beta)} (G \times H) = (T^*(G \times S) //_{\beta} H) //_{\alpha} G$$

as singular symplectic spaces.

Proof. Since the actions by λ and τ are free the first two assertions can be deduced from the regular commuting reduction theorem ([21]) with the necessary conditions being verified in the above lemma. Clearly, the product action by $G \times H$ is well-defined and Hamiltonian with asserted momentum map. However, the product action will not be free in general. Thus the last point is a consequence of the singular commuting reduction theorem of Section 3. \square

We will only be interested in the case where $\beta = 0$. Moreover, on T^*G we shall only be concerned with the lifted λ -action. Thus the expression $T^*G //_{\alpha} G$ will throughout stand for $(J^\lambda)^{-1}(\alpha) / G_\alpha$.

Proposition 4.3. *Clearly, 0 is in the image of J^τ . Therefore,*

$$\begin{aligned} T^*U //_{\alpha} G &\cong T^*(G \times_H S) //_{\alpha} G = T^*(G \times S) //_0 H //_{\alpha} G = T^*(G \times S) //_{\alpha} G //_0 H \\ &= (T^*G //_{\alpha} G \times T^*S) //_0 H = (\mathcal{O} \times T^*S) //_0 H \end{aligned}$$

as stratified symplectic spaces, and where $\mathcal{O} = \text{Ad}^(G).\alpha$.*

Proof. Since the isomorphism $T^*U \xrightarrow{\cong} T^*(G \times_H S)$ comes from an equivariant diffeomorphism $U \xrightarrow{\cong} G \times_H S$ on the base it is an equivariant symplectomorphism that intertwines the respective momentum maps. Now the regular reduction theorem for cotangent bundles at zero momentum says that $T^*(G \times_H S)$ and $T^*(G \times S) //_0 H$ are symplectomorphic. (See [1, Theorem 4.3.3] and the remark immediately below [1, Theorem 4.3.3].) Further it is well-known (and immediate from Lemma 4.1(6)) that $T^*G //_{\alpha} G = \mathcal{O}$. The rest is a direct consequence of Theorem 3.2 on singular commuting reduction. \square

From now on we make the assumption that

$$Q = Q_{(H)},$$

i.e., all isotropy subgroups of points $q \in Q$ are conjugate within G to H . Obviously, this assumption imposes a rather strong restriction on the generality of the subsequent. However, in applications such as in the Calogero-Moser system of Section 6 this is the generic case in a certain sense. See also Remark 4.5.

Theorem 4.4 (Bundle picture). *Let $Q = Q_{(H)}$ and let $\mathcal{O} \subseteq \mathfrak{g}^*$ be a coadjoint orbit in the image of the momentum map $\mu : T^*Q \rightarrow \mathfrak{g}^*$. Then, locally, we have a singular symplectic fiber bundle*

$$\mathcal{O} //_0 H \hookrightarrow T^*Q //_{\mathcal{O}} G \longrightarrow T^*(Q/G)$$

with typical fiber the singular symplectic space $\mathcal{O} //_0 H$ and smooth base $T^*(Q/G)$.

The fiber bundle in this theorem is singular in the sense that it is a topological fiber bundle and the transition functions act by strata preserving transformations on $\mathcal{O} //_0 H$ which are smooth in the sense that they preserve the algebra $C^\infty(\mathcal{O} //_0 H) := W^\infty(\mathcal{O} \cap \text{Ann } \mathfrak{h})^H$ where $W^\infty(\mathcal{O} \cap \text{Ann } \mathfrak{h})$ denotes the Whitney C^∞ functions on $\mathcal{O} \cap \text{Ann } \mathfrak{h} \subset \mathcal{O}$. See [4, 36, 37] for more on smooth structures on singular (symplectic) spaces.

Proof. Consider a tube U of the G -action on Q . By virtue of the Slice Theorem ([32, 11, 30]) there thus exists a slice S such that there is a G -equivariant diffeomorphism

$$U \cong G \times_H S = G/H \times S.$$

Indeed, this is true since all points of Q are regular by assumption whence the slice representation is trivial. We can lift this diffeomorphism to a symplectomorphism of cotangent bundles to get

$$T^*U //_{\mathcal{O}} G \cong \mathcal{O} //_0 H \times T^*S$$

as in Proposition 4.3 above. Since T^*S is a typical neighborhood in $T^*(Q/G)$ the result follows. \square

Remark 4.5 (On fully singular reduction). For the purpose of this remark assume that the isotropy lattice of the G -action on Q consists of more than one isotropy class. Let (H) be an isotropy class on Q , and let (L) be an isotropy class of the lifted G -action on T^*Q . Let $\text{Ann } Q_{(H)} \rightarrow Q_{(H)}$ denote the sub-bundle of $(T^*Q)|_{Q_{(H)}}$ consisting of those co-vectors which vanish upon insertion of a vector tangent to $Q_{(H)}$. Clearly, we have

$$(T^*Q)_{(L)}|_{Q_{(H)}} = (T^*Q_{(H)} \times_{Q_{(H)}} \text{Ann } Q_{(H)})_{(L)},$$

and note that the momentum map $\mu : T^*Q \rightarrow \mathfrak{g}^*$ vanishes on $\text{Ann } Q_{(H)}$. Therefore, for an orbit \mathcal{O} in the image of μ we have that

$$\mu^{-1}(\mathcal{O})|_{Q_{(H)}} = \mu_{(H)}^{-1}(\mathcal{O}) \times_{Q_{(H)}} \text{Ann } Q_{(H)}$$

where $\mu_{(H)}$ denotes the momentum map of the cotangent lifted G -action on $T^*Q_{(H)}$. The G -equivariant projection $\mu_{(H)}^{-1}(\mathcal{O}) \times_{Q_{(H)}} \text{Ann } Q_{(H)} \rightarrow \mu_{(H)}^{-1}(\mathcal{O})$ gives rise to a mapping

$$(F) \quad \eta_{(L)} : (\mu_{(H)}^{-1}(\mathcal{O}) \times_{Q_{(H)}} \text{Ann } Q_{(H)})_{(L)} / G \longrightarrow \mu_{(H)}^{-1}(\mathcal{O}) / G = T^*(Q_{(H)}) //_{\mathcal{O}} G$$

the base of which is described by Theorem 5.6 in the presence of a generalized connection form on $Q_{(H)} \rightarrow Q_{(H)}/G$. The map $\eta_{(L)}$ is, in general, neither surjective

nor does it have locally constant fiber type. The fiber over a point $[x] \in \mu_{(H)}^{-1}(\mathcal{O})/G$ (such that $G_{\tau(x)} = H$) is of the form

$$\eta_{(L)}^{-1}([x]) = \{w \in \text{Ann}_{\tau(x)} Q_{(H)} : H_w \cap G_x \text{ is conjugate to } L \text{ within } G\}/G_x$$

where $\tau : T^*Q_{(H)} \rightarrow Q_{(H)}$ is the cotangent projection. Note that $gLg^{-1} \subset G_x \subset G_{\tau(x)}$ for some $g \in G$ by equivariance of projections. The image of $\eta_{(L)}$ clearly is a union of orbit type strata. Moreover, using the notation of Duistermaat and Kolk [11, Definition 2.6.1] it is evident that

$$\text{im } \eta_{(L)} \subset (\mu_{(H)}^{-1}(\mathcal{O}))_{\tilde{x}}^{\leq}/G = G \cdot (\mu_{(H)}^{-1}(\mathcal{O}))^L/G$$

where $x \in (\mu_{(H)}^{-1}(\mathcal{O}))_{(L)} = (\mu_{(H)}^{-1}(\mathcal{O}))_{\tilde{x}}^{\sim}$, whence it follows that $(\mu_{(H)}^{-1}(\mathcal{O}))_{(L)}/G \subset \text{im } \eta_{(L)}$ is open and dense since it is the regular stratum of $\text{im } \eta_{(L)}$. Let $M_0 := (T^*(Q_{(H)})//\mathcal{O}G)_{(L)} := (\mu_{(H)}^{-1}(\mathcal{O}))_{(L)}/G$ (see Theorem 5.6) and consider the restriction $\eta_0 := \eta_{(L)}|_{\eta_{(L)}^{-1}(M_0)}$ which yields a bundle like object

$$\eta_0 : \eta_{(L)}^{-1}(M_0) \longrightarrow M_0,$$

that is, η_0 is surjective and the fiber over a point $[x] \in M_0$ such that $G_x = L$ is $(\text{Ann}_{\tau(x)} Q_{(H)})^L$. We believe that this object can be shown to constitute a smooth fiber bundle. However, the employability of this ‘bundle’ is quite limited by the fact that we cannot give a satisfactorily useful description of $\eta_{(L)}^{-1}(M_0)$. Further problems are deciding what a generalized connection form on $Q \rightarrow Q/G$ should be and determining how the ‘secondary strata’ $(\mu_{(H)}^{-1}(\mathcal{O}) \times_{Q_{(H)}} \text{Ann } Q_{(H)})_{(L)}/G$ fit together to yield the ‘primary stratum’ $(\mu_{(H)}^{-1}(\mathcal{O}) \cap (T^*Q)_{(L)})/G$. The latter problem was solved in [35] for reduction at trivial orbits $\mathcal{O} = \{\text{point}\}$.

If $Q_{(H)} = Q_{\text{reg}}$ is the regular stratum which is open dense in Q then $\text{Ann } Q_{(H)}$ is trivial. In this (generic) case Theorems 4.4 and 5.6 thus provide a full answer to the reduction problem. In the more general situation these results clearly provide only a partial answer to the reduction problem. However, it is expected that any solution to this problem will rely on these single orbit type results.

5. GAUGED COTANGENT BUNDLE REDUCTION

Continue to assume that we are in the situation of Section 4. In particular, we suppose that $Q = Q_{(H)}$ is of single orbit type. However, as an additional input datum we assume from now on a generalized principal bundle connection form $A \in \Omega^1(Q; \mathfrak{g})$ on $Q \rightarrow Q/G$ given. The term generalized is to be understood in the context of Alekseevsky and Michor [3, Section 3.1]. This means that $A : TQ \rightarrow \mathfrak{g}$ is G -equivariant and that $\zeta = \zeta \circ A \circ \zeta$. In particular, the connection form A induces a right inverse to the projection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{g}_q$ depending smoothly on $q \in Q$.

According to [3, Section 4.6] the curvature form associated to A is defined by

$$\text{Curv}^A := dA - \frac{1}{2}[A, A]^\wedge$$

where

$$[\varphi, \psi]^\wedge(v_1, \dots, v_{l+k}) := \frac{1}{k!l!} \sum_{\sigma} \text{sign } \sigma [\varphi(v_{\sigma_1}, \dots, v_{\sigma_l}), \psi(v_{\sigma_{l+1}}, \dots, v_{\sigma_{l+k}})]$$

is the graded Lie bracket on $\Omega(Q; \mathfrak{g}) := \bigoplus_{k=0}^{\infty} \Gamma(\Lambda^k T^*Q \otimes \mathfrak{g})$, and $\varphi \in \Omega^l(Q; \mathfrak{g})$ and $\psi \in \Omega^k(Q; \mathfrak{g})$. The sign in our definition of Curv^A differs from that in [3] because we are concerned with left G -actions as opposed to right actions.

Since the G -action on Q is of single orbit type the orbit space Q/G is a smooth manifold, and the projection $\pi : Q \rightarrow Q/G$ is a fiber bundle with typical fiber G/H . However, the isotropy lattice of the lifted action by G on T^*Q is, in general (for

$H \neq \{e\}$), non-trivial whence the quotient space $(T^*Q)/G$ is a stratified space. Its strata are of the form $(T^*Q)_{(L)}/G$ where (L) is in the isotropy lattice of T^*Q .

The vertical sub-bundle of TQ with respect to $\pi : Q \rightarrow Q/G$ is $\text{Ver} := \ker T\pi$. Via the connection A we can also define the horizontal sub-bundle $\text{Hor} := \ker A$. We define the dual horizontal sub-bundle of T^*Q as the sub-bundle Hor^* consisting of those co-vectors that vanish on all vertical vectors. Likewise, we define the dual vertical sub-bundle of T^*Q as the sub-bundle Ver^* consisting of those co-vectors that vanish on all horizontal vectors. As usual, the connection A provides a trivialization of the vertical sub-bundle, i.e., $\text{Ver} \cong_A \bigsqcup_{q \in Q} \mathfrak{g}/\mathfrak{g}_q$. In particular, $\bigsqcup_{q \in Q} \mathfrak{g}/\mathfrak{g}_q$ and $\bigsqcup_{q \in Q} \text{Ann } \mathfrak{g}_q$ are smooth vector bundles.

5.A. Mechanical connection. If $(Q, \langle \cdot, \cdot \rangle)$ is a Riemannian manifold and G acts on Q by isometries there is a certain connection which is particularly well adapted to mechanical systems on Q . This is the so-called mechanical connection which is defined as follows. For $X, Y \in \mathfrak{g}$ and $q \in Q$ we define $\mathbb{I}_q(X, Y) := \langle \zeta_X(q), \zeta_Y(q) \rangle$ and call this the LOCKED INERTIA TENSOR. This defines a non-degenerate pairing on $\mathfrak{g}/\mathfrak{g}_q$ whence it provides an identification $\check{\mathbb{I}}_q : \mathfrak{g}/\mathfrak{g}_q \rightarrow (\mathfrak{g}/\mathfrak{g}_q)^* = \text{Ann } \mathfrak{g}_q$. We use this isomorphism to define a one-form \check{A} on Q with values in the bundle $\bigsqcup_{q \in Q} \mathfrak{g}/\mathfrak{g}_q$ by the following diagram.

$$\begin{array}{ccc} T_q^*Q & \xrightarrow{\mu_q} & \text{Ann } \mathfrak{g}_q \\ \simeq \uparrow & & \downarrow (\check{\mathbb{I}}_q)^{-1} \\ T_qQ & \xrightarrow{\check{A}_q} & \mathfrak{g}/\mathfrak{g}_q \end{array}$$

Notice that $\text{im } \mu_q = \text{Ann } \mathfrak{g}_q$ by reason of dimension. Thus we have a trivialization $\text{Ver} \cong_{\check{A}} \bigsqcup_{q \in Q} \mathfrak{g}/\mathfrak{g}_q$ of the vertical sub-bundle. However, to obtain a generalized connection form on $Q \rightarrow Q/G$ from this trivialization we have to assume one additional object: namely, let $r_q : \mathfrak{g}/\mathfrak{g}_q \rightarrow \mathfrak{g}$ be a right inverse to the projection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{g}_q$ depending smoothly on $q \in Q$. The (generalized) MECHANICAL CONNECTION $A : TQ \rightarrow \mathfrak{g}$ on $Q \rightarrow Q/G$ is thus defined as the composition $A_q = r_q \circ \check{A}_q : T_qQ \rightarrow \mathfrak{g}/\mathfrak{g}_q \rightarrow \mathfrak{g}$.

One obvious way to obtain such a right inverse is to choose a G -invariant non-degenerate bilinear form on \mathfrak{g} such that $\mathfrak{g}/\mathfrak{g}_q \cong \mathfrak{g}_q^\perp \hookrightarrow \mathfrak{g}$ with respect to this form. In many examples such a non-degenerate form is given canonically.

The mechanical connection was first defined by Smale [42] for the case of Abelian group actions. See also Marsden, Montgomery, and Ratiu [22, Section 2].

To verify that the mechanical connection is indeed a generalized principal connection form one checks that $A : TQ \rightarrow \mathfrak{g}$ is equivariant and $\zeta(q)(A_q(\zeta_X(q))) = \zeta_X(q)$ for all $X \in \mathfrak{g}$.

5.B. Weinstein realization of T^*Q . Let A continue to denote a generalized principal connection form on Q , and let $\langle \cdot, \cdot \rangle$ denote the dual pairing. We define a point-wise dual $A_q^* : \text{Ann } \mathfrak{g}_q \rightarrow \text{Ver}_q^* \subseteq T_q^*Q$ by the formula $\langle A_q^*(\lambda), v \rangle = \langle \lambda, A_q(v) \rangle$ where $\lambda \in \text{Ann } \mathfrak{g}_q$ and $v \in T_qQ$. Notice that $A_q^*(\mu_q(p)) = p$ for all $p \in \text{Ver}_q^*$ and $\mu_q(A_q^*(\lambda)) = \lambda$ for all $\lambda \in \text{Ann } \mathfrak{g}_q$ since A is a connection form.

Let $\pi : Q \rightarrow Q/G$ and $\tau_Q : TQ \rightarrow Q$ denote the projections. From the connection form A we obtain the horizontal lift mapping which we denote by

$$C := ((\tau_Q, T\pi)|_{\text{Hor}})^{-1} : Q \times_{Q/G} T(Q/G) \rightarrow \text{Hor} \hookrightarrow TQ.$$

Its fiber restriction shall be denoted by $C_q : \{q\} \times T_{\pi(q)}(Q/G) \rightarrow \text{Hor}_q \hookrightarrow T_qQ$.

Using the horizontal lift C on the one hand and the connection A on the other hand we obtain a G -equivariant isomorphism

$$TQ = \text{Hor} \oplus \text{Ver} \longrightarrow (Q \times_{Q/G} T(Q/G)) \times_Q \bigsqcup_{q \in Q} \mathfrak{g}/\mathfrak{g}_q$$

of bundles over Q . There is a dual version to this isomorphism, and we will abbreviate

$$\mathcal{W} := (Q \times_{Q/G} T^*(Q/G)) \times_Q \bigsqcup_{q \in Q} \text{Ann } \mathfrak{g}_q \cong \text{Hor}^* \oplus \text{Ver}^*.$$

The explicit form of the isomorphism $\mathcal{W} \cong T^*Q$ is stated in Proposition 5.1 below. To set up some notation for the upcoming proposition, and clarify the picture consider the following stacking of pull-back diagrams.

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\rho^* \tilde{\tau} = \tilde{\tilde{\tau}}} & \bigsqcup_q \text{Ann } \mathfrak{g}_q \\ \tilde{\tau}^* \rho = \tilde{\rho} \downarrow & & \downarrow \rho \\ Q \times_{Q/G} T^*(Q/G) & \xrightarrow{\pi^* \tau = \tilde{\tau}} & Q \\ \tau^* \pi = \tilde{\pi} \downarrow & & \downarrow \pi \\ T^*(Q/G) & \xrightarrow{\tau} & Q/G \end{array}$$

The upper stars in this diagram are, of course, not pull-back stars. By slight abuse of notation we shall denote elements $(q; \pi(q), \eta; q, \lambda) \in \mathcal{W}$ simply by (q, η, λ) . Further, let $\tau_{\mathcal{W}} : T\mathcal{W} \rightarrow \mathcal{W}$ denote the tangent projection.

Proposition 5.1 (Symplectic structure on \mathcal{W}). *The chosen connection form A induces an isomorphism*

$$\begin{aligned} \psi = \psi(A) : (Q \times_{Q/G} T^*(Q/G)) \times_Q \bigsqcup_{q \in Q} \text{Ann } \mathfrak{g}_q = \mathcal{W} &\longrightarrow T^*Q, \\ (q, \eta, \lambda) &\longmapsto (q, (T_q \pi)^* \eta + A_q^*(\lambda)) \end{aligned}$$

of bundles over Q , and the following are true.

- (1) *There is an induced G -action on $(\mathcal{W}, \sigma = \psi^* \Omega)$ by symplectomorphisms. Here $\Omega = -d\theta$ is the canonical symplectic form on T^*Q . Moreover, this action is Hamiltonian with momentum map*

$$\mu_A = \mu \circ \psi : \mathcal{W} \longrightarrow \mathfrak{g}^*, (q, \eta, \lambda) \longmapsto \lambda,$$

where μ is the momentum map $T^*Q \rightarrow \mathfrak{g}^*$, and ψ is equivariant.

- (2) *The induced symplectic form σ on the connection dependent realization \mathcal{W} of T^*Q is given by the formula*

$$\sigma = (\tilde{\pi} \circ \tilde{\rho})^* \Omega^{Q/G} - dB.$$

Here $\Omega^{Q/G} = -d\theta^{Q/G}$ is the canonical symplectic form on $T^*(Q/G)$, and $B \in \Omega^1(\mathcal{W})$ is given by

$$B = \langle \tau_{\mathcal{W}}^* \mu_A, (\tilde{\tau} \circ \tilde{\rho})^* A \rangle.$$

Moreover,

$$\begin{aligned} dB &= \langle \tau_{\mathcal{W}}^* d\mu_A \wedge (\tilde{\tau} \circ \tilde{\rho})^* A \rangle + \langle \tau_{\mathcal{W}}^* \mu_A, (\tilde{\tau} \circ \tilde{\rho})^* \text{Curv}^A \rangle \\ &\quad + \langle \tau_{\mathcal{W}}^* \mu_A, \frac{1}{2} [(\tilde{\tau} \circ \tilde{\rho})^* A, (\tilde{\tau} \circ \tilde{\rho})^* A] \wedge \rangle. \end{aligned}$$

Here $\langle \cdot \wedge \cdot \rangle$ denotes the exterior multiplication of a \mathfrak{g}^* -valued form with a \mathfrak{g} -valued form. In local coordinates where we may use a splitting of tangent vectors $\xi_1, \xi_2 \in T_{(q, \eta, \lambda)} \mathcal{W}$ as $\xi_i = (q'_i, \eta'_i, \lambda'_i)$ and $q'_i = v_i^{\text{hor}} + \zeta_{Z_i}(q)$ for $i = 1, 2$ this means that

$$dB_{(q, \eta, \lambda)}(\xi_1, \xi_2) = \langle \lambda'_1, Z_2 \rangle - \langle \lambda'_2, Z_1 \rangle + \langle \lambda, \text{Curv}_q^A(q'_1, q'_2) \rangle + \langle \lambda, [Z_1, Z_2] \rangle.$$

Proof. Obviously, ψ is a bundle map. Its inverse is given by the bundle map $(q, p) \mapsto (q, C_q^*(p), \mu_q(p))$ where $C_q : \{q\} \times T_{\pi(q)}(Q/G) \rightarrow T_q Q$ is the horizontal lift mapping and $\mu_q := \mu|_{T_q^* Q}$ is the fiber restriction of the momentum map.

Assertion (1) is clear from the construction.

To see assertion (2) we work locally in Q . That is, let U be a trivializing patch for $\mathcal{W} \rightarrow Q$ as well as for $T^*Q \rightarrow Q$ and consider $\psi|_U : \mathcal{W}|_U \rightarrow T^*Q|_U$. Let $(q, \eta, \lambda) \in \mathcal{W}|_U$ and $\xi \in T_{(q, \eta, \lambda)}\mathcal{W}$. By locality we may split ξ as $\xi = (q', \eta', \lambda')$ where $q' \in T_q Q$, $\eta' \in T_\eta(T^*(Q/G))$, and $\lambda' \in T_\lambda(\bigsqcup_{q \in Q} \text{Ann } \mathfrak{g}_q)$. However, in order to make the notation not too cumbersome we do not invent new symbols (like ξ^U or A^U) for the local versions of global objects (like ξ or A). To find the desired formula for $\sigma = \psi^* \Omega = -d\psi^* \theta$, note that

$$\begin{aligned} (\psi^* \theta)_{(q, \eta, \lambda)}(\xi) &= \theta_{(q, (T_q \pi)^* \eta + A_q^*(\lambda))}(T_{(q, \eta, \lambda)} \psi \cdot \xi) \\ &= \theta_{(q, (T_q \pi)^* \eta + A_q^*(\lambda))}(T_{(q, \eta, \lambda)} \psi^1 \cdot \xi, T_{(q, \eta, \lambda)} \psi^2 \cdot \xi) \\ &= \langle (T_q \pi)^* \eta + A_q^*(\lambda), T_{(q, (T_q \pi)^* \eta + A_q^*(\lambda))} \tau_Q \cdot T_{(q, \eta, \lambda)} \psi^1 \cdot \xi \rangle \\ &= \langle (T_q \pi)^* \eta, q' \rangle + \langle A_q^*(\lambda), q' \rangle \\ &= \langle \eta, T_q \pi \cdot q' \rangle + \langle \lambda, A_q(q') \rangle \\ &= ((\tilde{\pi} \circ \tilde{\rho})^* \theta^{Q/G})_{(q, \eta, \lambda)}(\xi) + (\langle \tau_{\mathcal{W}}^* \mu_A, (\tilde{\tau} \circ \tilde{\rho})^* A \rangle)_{(q, \eta, \lambda)}(\xi) \\ &= (((\tilde{\pi} \circ \tilde{\rho})^* \theta^{Q/G}) + B)_{(q, \eta, \lambda)}(\xi) \end{aligned}$$

where $\psi|_U = (\psi^1, \psi^2) : \mathcal{W}|_U \rightarrow U \times V = T^*Q|_U$ and V is the standard fiber of $\tau_Q : T^*Q \rightarrow Q$. Since the first and the last expressions in this computation are global objects it is true that

$$\begin{aligned} \sigma &= (\tilde{\pi} \circ \tilde{\rho})^* \Omega^{Q/G} - dB \\ &= (\tilde{\pi} \circ \tilde{\rho})^* \Omega^{Q/G} - \langle \tau_{\mathcal{W}}^* d\mu_A \wedge (\tilde{\tau} \circ \tilde{\rho})^* A \rangle \\ &\quad - \langle \tau_{\mathcal{W}}^* \mu_A, (\tilde{\tau} \circ \tilde{\rho})^* \text{Curv}^A \rangle + \langle \tau_{\mathcal{W}}^* \mu_A, \frac{1}{2} [(\tilde{\tau} \circ \tilde{\rho})^* A, (\tilde{\tau} \circ \tilde{\rho})^* A]^\wedge \rangle. \end{aligned}$$

Indeed, this is because $\text{Curv}^A = dA - \frac{1}{2} [A, A]^\wedge$. \square

The G -action on \mathcal{W} is, of course, given by $g \cdot (q, \eta, \lambda) = (g \cdot q, \eta, \text{Ad}^*(g) \cdot \lambda)$. Similarly there is an induced G -action on $\bigsqcup_{q \in Q} \mathfrak{g}/\mathfrak{g}_q \cong \text{Ver}$ which is given by $g \cdot (q, X + \mathfrak{g}_q) = (g \cdot q, \text{Ad}(g) \cdot X + \mathfrak{g}_{g \cdot q})$.

The notion of a stratified map which appears in the following theorem is defined in Section 2.

Theorem 5.2 (Weinstein space). *There are stratified isomorphisms of stratified bundles over Q/G :*

$$\begin{aligned} \alpha = \alpha(A) : \bigsqcup_{(L)} (TQ)_{(L)}/G &\longrightarrow T(Q/G) \times_{Q/G} \bigsqcup_{(L)} (\bigsqcup_{q \in Q} \mathfrak{g}/\mathfrak{g}_q)_{(L)}/G, \\ [(q, v)] &\longmapsto (T\pi(q, v), [(q, A_q(v))]) \end{aligned}$$

where (L) runs through the isotropy lattice of TQ . The dual isomorphism is given by

$$\begin{aligned} \beta = (\alpha^{-1})^* : (T^*Q)/G &\longrightarrow T^*(Q/G) \times_{Q/G} (\bigsqcup_{q \in Q} \text{Ann } \mathfrak{g}_q)/G =: W, \\ [(q, p)] &\longmapsto (C_q^*(p), [(q, \mu(q, p))]) \end{aligned}$$

where the stratification was suppressed. Here

$$C_q^* : T_q^* Q \xrightarrow{\iota_q^*} \text{Hor}_q^* \rightarrow T_{\pi(q)}^*(Q/G)$$

is the point wise dual to the horizontal lift mapping

$$C : Q \times_{Q/G} T(Q/G) \longrightarrow \text{Hor} \xrightarrow{\iota_q} TQ, ([q], v; q) \longmapsto C_q(v).$$

Moreover, β is an isomorphism of Poisson spaces as follows: we can naturally identify

$$\mathcal{W}/G \xrightarrow{\cong} W, [(q; [q], \eta; q, \lambda)] \longmapsto ([q], \eta; [(q, \lambda)])$$

thus obtaining a quotient Poisson bracket on $C^\infty(W) = C^\infty(\mathcal{W})^G$ as the quotient Poisson bracket.

In the case that G acts on Q freely the first assertion of the above theorem can also be found in Cendra, Holm, Marsden, Ratiu [8]. Following Ortega and Ratiu [30, Section 6.6.12] the above constructed interpretation W of $(T^*Q)/G$ is called WEINSTEIN SPACE referring to Weinstein [47] where this universal construction first appeared.

In fact, the original construction of [47] was the following: Let Q be a left free and proper G -space such that $Q \rightarrow Q/G$ is endowed with a principal bundle connection form A and let F be a right Hamiltonian G -space with equivariant momentum map $\Phi : F \rightarrow \mathfrak{g}^*$. To make F into a left Hamiltonian G -space we use the inversion in the group. The momentum map with respect to the thus obtained G -action is given by $-\Phi$. Under these assumptions [47] proves that the smooth symplectic quotient $(T^*Q \times F) //_0 G = (\mu - \Phi)^{-1}(0)/G \cong_A T^*(Q/G) \times_{Q/G} (Q \times_G F)$ is symplectomorphic to the Sternberg space $(Q \times_{Q/G} T^*(Q/G)) \times_G F$ of [43]. Taking F to be the Hamiltonian G -space \mathcal{O} acted upon by $\text{Ad}^*(G)$, and employing the shifting trick $T^*Q //_{\mathcal{O}} G \cong (T^*Q \times \mathcal{O}) //_0 G$, this construction yields the realization $T^*Q //_{\mathcal{O}} G \cong_A T^*(Q/G) \times_{Q/G} (Q \times_G \mathcal{O})$. In particular, one thus obtains a fiber bundle $\mathcal{O} \hookrightarrow T^*Q //_{\mathcal{O}} G \rightarrow T^*(Q/G)$.

It thus makes sense to refer to the realization W of $(T^*Q)/G$ which is constructed along similar lines as a Weinstein space. The induced Poisson structure on W is explicitly described in [16].

Proof. As already noted above $(TQ)/G$ is a stratified space. Since the base Q is stratified as consisting only of a single stratum, the equivariant foot point projection map $\tau : TQ \rightarrow Q$ is trivially a stratified map. Thus, we really get a stratified bundle $(TQ)/G \rightarrow Q/G$. In the same spirit $(\bigsqcup_{q \in Q} \mathfrak{g}/\mathfrak{g}_q)/G$ is stratified into orbit types, and the projection onto Q/G is a stratified bundle map. According to Davis [10] pullbacks are well defined in the category of stratified spaces and stratified maps and thus it makes sense to define $T(Q/G) \times_{Q/G} (\bigsqcup_{q \in Q} \mathfrak{g}/\mathfrak{g}_q)/G$.

The map α is well defined: indeed, for $(q, v) \in TQ$ and $k \in G$ we have $T\pi(k.q, k.v) = (\pi(k.q), T_{k.q}\pi(T_q l_k(v))) = (\pi(q), T_q(\pi \circ l_k)(v)) = T\pi(q, v)$, and $[(k.q, A_{k.q}(k.v))] = [(q, A_q(v))]$ by equivariance of A . It is clearly continuous as a composition of continuous maps.

We claim that α maps strata onto strata, and moreover we have the formula

$$\alpha((TQ)_{(L)}/G) = T(Q/G) \times_{Q/G} (\bigsqcup_{q \in Q} \mathfrak{g}/\mathfrak{g}_q)_{(L)}/G.$$

Indeed, this follows immediately since α lifts to a smooth equivariant isomorphism $\tilde{\alpha} : TQ \rightarrow (Q \times_{Q/G} T(Q/G)) \times_Q \bigsqcup_{q \in Q} \mathfrak{g}/\mathfrak{g}_q$, $(q, v) \mapsto (q; T\pi(q, v); q, A_q(v))$ of vector bundles over Q , and we have clearly that $\tilde{\alpha}((TQ)_{(L)}) = (Q \times_{Q/G} T(Q/G)) \times_Q (\bigsqcup_{q \in Q} \mathfrak{g}/\mathfrak{g}_q)_{(L)}$. The restriction of α to any stratum is smooth as a composition of smooth maps.

Since $\zeta(q)(A_q(\zeta_X(q))) = \zeta_X(q)$ for $X \in \mathfrak{g}$ we can write down an inverse

$$\alpha^{-1} : ([q], v; [(q, X)]) \mapsto [(q, C_q(v) + \zeta_X(q))]$$

and again it is an easy matter to notice that this map is well defined, continuous, and smooth on each stratum. In fact, we have used here that the connection A by

definition provides a right inverse to $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{g}_q$ whence by slight abuse of notation we may consider elements $X \in \mathfrak{g}/\mathfrak{g}_q$ as elements in \mathfrak{g} .

It makes sense to define the dual β of the inverse map α^{-1} in a point wise manner, and it only remains to compute this map.

$$\begin{aligned} \langle \beta[(q, p)], ([q], v; [(q, X)]) \rangle &= \langle [(q, p)], [(q, C_q(v) + \zeta_X(q))] \rangle \\ &= \langle p, C_q(v) \rangle + \langle p, \zeta_X(q) \rangle \\ &= \langle C_q^*(p), v \rangle + \langle \mu(q, p), X \rangle \\ &= \langle (C_q^*(p), [(q, \mu(q, p))]), ([q], v; [(q, X)]) \rangle \end{aligned}$$

where we used the G -invariance of the dual pairing over Q .

Finally, β is an isomorphism of Poisson spaces: note first that the identifying map $\mathcal{W}/G \rightarrow W$, $[(q; [q], \eta; q, \lambda)] \mapsto ([q], \eta; [(q, \lambda)])$ is well-defined because G_q acts trivially on $\text{Hor}_q^* \cong T_{[q]}^*(Q/G) \ni \eta$ which in turn is due to the fact that all points of Q are regular. The quotient Poisson bracket is well-defined since $C^\infty(\mathcal{W})^G \subseteq C^\infty(W)$ is a Poisson sub-algebra. The statement now follows because the diagram

$$\begin{array}{ccc} T^*Q & \xrightarrow{\psi^{-1}} & \mathcal{W} \\ \downarrow & & \searrow \\ (T^*Q)/G & \xrightarrow{\beta} & W \equiv \mathcal{W}/G \end{array}$$

is commutative, and composition of top and down-right arrow is Poisson and the left vertical arrow is surjective. \square

5.C. The reduced phase space. The following lemmas are key to the subsequent. They guarantee, in particular, that every non-empty pre-image of a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ under μ fibrates surjectively over Hor^* . In Theorem 5.6 we use this to show that every non-empty symplectic stratum of the reduced phase space fibrates over $T^*(Q/G)$. The meta principle motivating these results is that μ (which is defined by means of the universal connection θ) can be thought of as a kind of universal connection form on $Q \rightarrow Q/G$.

Lemma 5.3. *Let $\mathcal{O} \subseteq \mathfrak{g}^*$ be a coadjoint orbit, $\mu : T^*Q \rightarrow \mathfrak{g}^*$ the canonical momentum mapping, and $\mu_q := \mu|_{(T_q^*Q)}$. Then either $\mu_q^{-1}(\mathcal{O}) = \emptyset$ for all $q \in Q$ or $\mu_q^{-1}(\mathcal{O}) \neq \emptyset$ for all $q \in Q$. In the latter case we have*

$$\mu_q^{-1}(\mathcal{O}) = \text{Ann}_q(T_q(G \cdot q)) \times \{A_q^*(\lambda) : \lambda \in \text{Ann } \mathfrak{g}_q \cap \mathcal{O}\}$$

which is an equality of topological spaces and where $A_q^* : \text{Ann } \mathfrak{g}_q \rightarrow \text{Ver}_q^*$ is the adjoint of $A_q : T_qQ \rightarrow \mathfrak{g}/\mathfrak{g}_q$.

Proof. Assume firstly that $q_1, q_2 \in Q$ lie in the same G -orbit. Then it is obviously true that $\mu_{q_1}^{-1}(\mathcal{O})$ is empty if and only if $\mu_{q_2}^{-1}(\mathcal{O})$ is empty. Thus for the purpose of this proof we can assume that $G_{q_1} = G_{q_2} = H$: all isotropy subgroups are conjugate to each other, and q_2 can be moved around in its orbit without loss of generality. Now assume that $\mu_{q_1}^{-1}(\mathcal{O})$ is non-empty, i.e., there is $\lambda = \mu_{q_1}(p_1) \in \text{Ann } \mathfrak{h} \cap \mathcal{O}$. Using the connection A we may then define $p_2 := A_{q_2}^*(\lambda) \in \text{Ver}_{q_2}^* \cap \mu_{q_2}^{-1}(\mathcal{O})$ whence $\mu_{q_2}^{-1}(\mathcal{O})$ is non-empty as well. In fact, this construction also proves the last claim of the lemma. \square

Lemma 5.4. *Let \mathcal{O} be a coadjoint orbit in the image of the momentum map $\mu_A : \mathcal{W} \rightarrow \mathfrak{g}^*$. Further, let (L) be in the isotropy lattice of the G -action on \mathcal{W} such that $\mu_A^{-1}(\mathcal{O}) \cap \mathcal{W}_{(L)} \neq \emptyset$. Then*

$$\mathcal{W}_{(L)} = (Q \times_{Q/G} T^*(Q/G)) \times_Q (\bigsqcup_{q \in Q} \text{Ann } \mathfrak{g}_q)_{(L)}$$

and

$$\mathcal{W}_{(L)} \cap \mu_A^{-1}(\mathcal{O}) = (Q \times_{Q/G} T^*(Q/G)) \times_Q (\bigsqcup_{q \in Q} \text{Ann } \mathfrak{g}_q \cap \mathcal{O})_{(L)}$$

are smooth manifolds. Moreover,

$$\mathcal{O}_{(L_0)_H^G} \cap \text{Ann } \mathfrak{h} \hookrightarrow (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{g}_q)_{(L)} \longrightarrow Q$$

is a smooth fiber bundle where L_0 is a subgroup of H such that L_0 is conjugate to L within G .

Notice that we do not assume $\mathcal{O} \cap \text{Ann } \mathfrak{h}$ to be smooth. The notation $\mathcal{O}_{(L_0)_H^G}$ is explained in Section 2(3). The G -action on $\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{g}_q$ is induced from the G -action on \mathcal{W} from Proposition 5.1, and is given by $g.(q, \lambda) = (g.q, \text{Ad}^*(g).\lambda)$.

Proof. The statement about $\mathcal{W}_{(L)}$ is clear. Thus also the description of $\mathcal{W}_{(L)} \cap \mu_A^{-1}(\mathcal{O})$ follows from the previous lemma together with Theorem 3.1.

Concerning the second assertion let $q_0 \in Q$ with $G_{q_0} = H$. Then

$$(q_0, \lambda) \in (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{g}_q)_{(L)}$$

if and only if

$$\lambda \in \mathcal{O} \cap \text{Ann } \mathfrak{h} \text{ and } H \cap G_\lambda = H_\lambda = L_0 \text{ is conjugate to } L \text{ in } G$$

which is true if and only if

$$\lambda \in (\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)_H^G}$$

where L_0 is a subgroup of H conjugate to L within G , and we view $\mathcal{O} \cap \text{Ann } \mathfrak{h} \subset \mathcal{O}$ as an H -space by virtue of the restricted $\text{Ad}^*(H)$ -action. By Lemma 5.5 the space $(\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)_H^G}$ is a smooth manifold.

To see smooth local triviality we proceed as follows. Let again $q_0 \in Q$ with $G_{q_0} = H$, and let S be a slice at q_0 and U a tube around $G.q_0$. That is, $G/H \times S \cong U$, $(kH, s) \mapsto k.s$ as proper G -spaces by virtue of the Slice Theorem ([32, 11]). Then we consider the smooth trivializing map

$$\begin{aligned} S \times G \times_H (\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)_H^G} &\longrightarrow (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{g}_q)_{(L)}|_U, \\ (s, [(k, \lambda_0)]_H) &\longmapsto (k.s, \text{Ad}^*(k).\lambda_0) \end{aligned}$$

which is well defined since $[(k, \lambda_0)]_H \in G \times_H (\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)_H^G}$ implies that $H_{\lambda_0} = gL_0g^{-1} \in (L)$ for some $g \in G$ and this yields $G_{(k.s, \text{Ad}^*(k).\lambda_0)} = kG_{(s, \lambda_0)}k^{-1} = kH_{\lambda_0}k^{-1} = kgL_0g^{-1}k^{-1}$. Hereby we use, firstly, that the diagonal H -action cancels out, i.e., $\text{Ad}^*(kh^{-1}).\text{Ad}^*(h).\lambda_0 = \text{Ad}^*(k).\lambda_0$ for all $h \in H$; secondly, we use that $\mathfrak{g}_s = \mathfrak{g}_{q_0} = \mathfrak{h}$ for all $s \in S$ since S is a slice at q_0 – see, e.g., [32, Corollary 5.1.13(2)]. Clearly, this map is smooth with smooth inverse $(q, \lambda) = (k.s, \text{Ad}^*(k).\lambda_0) \mapsto (s, [(k, \lambda_0)]_H)$. Therefore, this construction provides smooth bundle charts of the total space $(\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{g}_q)_{(L)}$. \square

Lemma 5.5. *Let G, \mathcal{O}, H and L_0 be as in Lemma 5.4. Then $(\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)_H^G}$ is a smooth (possibly disconnected) pre-symplectic manifold.*

This fact is an instance of the general theory of singular commuting reduction of [21]. However, the argument is rather explicit.

Proof. As in Lemma 4.1 consider the G -action on $T^*G = G \times \mathfrak{g}^*$ given by $g.(k, \lambda) = (gk, \lambda)$ and the H -action given by $h.(k, \lambda) = (kh^{-1}, \text{Ad}^*(h).\lambda)$. These actions are Hamiltonian. By Lemma 4.1(7) symplectic reduction of $G \times \mathfrak{g}^*$ with respect to H at $0 \in \mathfrak{h}^*$ yields $(G \times \mathfrak{g}^*)//_0 H = G \times_H \text{Ann } \mathfrak{h}$. Let $\pi_H : G \times \text{Ann } \mathfrak{h} \rightarrow G \times_H \text{Ann } \mathfrak{h}$ denote the orbit projection. The corollary of Lemma 4.1 (or straightforward computation) implies that there is an induced G -action on $G \times_H \text{Ann } \mathfrak{h}$ given by $g.[(k, \lambda)]_H =$

$[(gk, \lambda)]_H$ with momentum map j given by $j[(k, \lambda)]_H = \text{Ad}^*(k).\lambda$. (The formula for j follows from Lemma 4.1(6).) Now the theory of singular symplectic reduction (see Theorem 3.1) implies that

$$(j^{-1}(\mathcal{O}))_{(L_0)^G} = j^{-1}(\mathcal{O}) \cap (G \times_H \text{Ann } \mathfrak{h})_{(L_0)^G} = \\ \{[(k, \lambda)]_H \in G \times_H (\mathcal{O} \cap \text{Ann } \mathfrak{h}) : G_{[(k, \lambda)]_H} = kH\lambda k^{-1} \text{ is conjugate to } L_0 \text{ in } G\}$$

is a sub-manifold of $G \times_H \text{Ann } \mathfrak{h}$. Therefore, the pre-image $\pi_H^{-1}((j^{-1}(\mathcal{O}))_{(L_0)^G}) = G \times (\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)^G}$ is a sub-manifold of $G \times \text{Ann } \mathfrak{h}$, and $(\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)^G}$ is a sub-manifold of $\text{Ann } \mathfrak{h}$.

Finally, the manifold in question is pre-symplectic since $(\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)^G}/H \cong (G \times_H (\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)^G})/G = (j^{-1}(\mathcal{O}))_{(L_0)^G}/G$ is a symplectic manifold. \square

The singular reduction diagram of Ortega and Ratiu [30, Theorem 8.4.4] adjoined to the universal reduction procedure of Arms, Cushman, and Gotay [4] (see also [30, Section 10.3.2]) applied to the Weinstein space has the following form.

$$\begin{array}{ccccc} \mu_A^{-1}(\mathcal{O}) & \longleftarrow & \mu_A^{-1}(\lambda) & \hookrightarrow & \mathcal{W} \\ \downarrow & & \downarrow & & \downarrow \\ \mu_A^{-1}(\mathcal{O})/G & \xleftarrow{\cong} & \mu_A^{-1}(\lambda)/G & \hookrightarrow & \mathcal{W}/G =: W \end{array}$$

where $\lambda \in \mu_A(\mathcal{W})$ and \mathcal{O} is the coadjoint orbit passing through λ . Therefore it is a sensible generalization of the smooth case to interpret the reduced space $\mu_A^{-1}(\mathcal{O})/G = \mathcal{W} //_{\mathcal{O}} G$ as a typical stratified symplectic leaf of the stratified Poisson space W . The following thus generalizes the result of Marsden and Perlmutter [23, Theorem 4.3] to the case of a non-free but single orbit type action of G on Q .

Let \mathcal{O} be a coadjoint orbit in the image of the momentum map $\mu_A : \mathcal{W} \rightarrow \mathfrak{g}^*$, and let (L) be in the isotropy lattice of the G -action on \mathcal{W} such that $\mathcal{W}_{(L)}^{\mathcal{O}} := \mu_A^{-1}(\mathcal{O}) \cap \mathcal{W}_{(L)} \neq \emptyset$. Then we define

$$\iota_{(L)}^{\mathcal{O}} : \mathcal{W}_{(L)}^{\mathcal{O}} \hookrightarrow \mathcal{W},$$

the canonical embedding, and the orbit projection mapping

$$\pi_{(L)}^{\mathcal{O}} : \mathcal{W}_{(L)}^{\mathcal{O}} \twoheadrightarrow \mathcal{W}_{(L)}^{\mathcal{O}}/G =: (\mathcal{W} //_{\mathcal{O}} G)_{(L)}.$$

Further, we denote the Kirillov-Kostant-Souriau symplectic form on \mathcal{O} by $\Omega^{\mathcal{O}}$, that is $\Omega^{\mathcal{O}}(\lambda)(\text{ad}^*(X).\lambda, \text{ad}^*(Y).\lambda) = \langle \lambda, [X, Y] \rangle$. Remember from Proposition 5.1 that the symplectic structure on $\mathcal{W} = \mathcal{W}(A)$ is denoted by $\sigma = \psi^*\Omega$.

Theorem 5.6 (Gauged symplectic reduction). *Let $Q = Q_{(H)}$ and let A be a generalized connection form on $\pi : Q \twoheadrightarrow Q/G$. Let \mathcal{O} be a coadjoint orbit in the image of the momentum map $\mu_A : \mathcal{W} \rightarrow \mathfrak{g}^*$, and let (L) be in the isotropy lattice of the G -action on \mathcal{W} such that $\mathcal{W}_{(L)}^{\mathcal{O}} := \mu_A^{-1}(\mathcal{O}) \cap \mathcal{W}_{(L)} \neq \emptyset$. Then the following are true.*

- (1) *The smooth manifold $(\mathcal{W} //_{\mathcal{O}} G)_{(L)}$ is a typical symplectic stratum of the singular symplectic space $\mathcal{W} //_{\mathcal{O}} G$. The smooth symplectic manifold*

$$(\mathcal{O} //_{\mathcal{O}} H)_{(L_0)^G} := (\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)^G}/H,$$

where L_0 is an isotropy subgroup of the H -action on $\mathcal{O} \cap \text{Ann } \mathfrak{h}$, is a disjoint union of typical smooth symplectic strata of the singular symplectic space $\mathcal{O} //_{\mathcal{O}} H$.

(2) The symplectic stratum $(\mathcal{W} //_{\mathcal{O}} G)_{(L)}$ can be globally described as

$$(\mathcal{W} //_{\mathcal{O}} G)_{(L)} = T^*(Q/G) \times_{Q/G} (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{g}_q)_{(L)}/G$$

whence it is the total space of the smooth symplectic fiber bundle

$$(\mathcal{O} //_{\mathcal{O}} H)_{(L_0)_{\mathbb{H}}} \xrightarrow{\subset} (\mathcal{W} //_{\mathcal{O}} G)_{(L)} \xrightarrow{\chi} T^*(Q/G).$$

Hereby $L_0 \subset H$ is an isotropy subgroup of the induced H -action on \mathcal{O} which is conjugate to L within G .

(3) The symplectic structure $\sigma_{(L)}^{\mathcal{O}}$ on $(\mathcal{W} //_{\mathcal{O}} G)_{(L)}$ is uniquely determined and given by the formula

$$(\pi_{(L)}^{\mathcal{O}})^* \sigma_{(L)}^{\mathcal{O}} = (\iota_{(L)}^{\mathcal{O}})^* \sigma - (\mu_A | \mathcal{W}_{(L)}^{\mathcal{O}})^* \Omega^{\mathcal{O}}.$$

Therefore,

$$\sigma_{(L)}^{\mathcal{O}} = \chi^* \Omega^{Q/G} - \beta_{(L)}^{\mathcal{O}}$$

where $\beta_{(L)}^{\mathcal{O}} \in \Omega^2((\mathcal{W} //_{\mathcal{O}} G)_{(L)})$ is defined by

$$(\pi_{(L)}^{\mathcal{O}})^* \beta_{(L)}^{\mathcal{O}} = (\iota_{(L)}^{\mathcal{O}})^* dB + (\mu_A | \mathcal{W}_{(L)}^{\mathcal{O}})^* \Omega^{\mathcal{O}}.$$

Finally, B is the form that was introduced in Proposition 5.1.

(4) The stratified symplectic space can be globally described as

$$\mathcal{W} //_{\mathcal{O}} G = T^*(Q/G) \times_{Q/G} \bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{g}_q / G$$

whence it is the total space of

$$\mathcal{O} //_{\mathcal{O}} H \xrightarrow{\subset} \mathcal{W} //_{\mathcal{O}} G \longrightarrow T^*(Q/G)$$

which is a stratified symplectic fiber bundle with singularities confined to the fiber direction.

Proof. Assertion (1). This is an implication of the general theory of stratified symplectic reduction – see Ortega and Ratiu [30, Section 8.4] or Section 3 for a statement of the relevant theorem and Section 2 for the notation.

To see that $(\mathcal{O} //_{\mathcal{O}} H)_{(L_0)_{\mathbb{H}}}$ is a union of typical smooth symplectic strata of the singular symplectic space $\mathcal{O} //_{\mathcal{O}} H$ note firstly that $(\mathcal{O} //_{\mathcal{O}} H)_{(L_0)_{\mathbb{H}}}$ is, according to the proof of Lemma 5.5, a typical stratum of $(T^*G) //_{\mathcal{O}} H //_{\mathcal{O}} G$. By the corollary of Lemma 4.1 (with $S = \{\text{point}\}$ and $\beta = 0$) there is an isomorphism $(T^*G) //_{\mathcal{O}} H //_{\mathcal{O}} G \cong (T^*G) //_{\mathcal{O}} G //_{\mathcal{O}} H = \mathcal{O} //_{\mathcal{O}} H$, $[(g, \lambda)]_H \mapsto [(g, \lambda)]_G \mapsto [\lambda]_H$ of singular symplectic spaces, whence strata are mapped symplectomorphically onto unions of strata.

Assertion (2). The description of the stratum $(\mathcal{W} //_{\mathcal{O}} G)_{(L)}$ follows from Proposition 5.1.

We know from (1) that all spaces involved in the diagram really are smooth. As in the proof of Lemma 5.4 let $q_0 \in Q$ with $G_{q_0} = H$, S a slice at q_0 , and $U \cong G/H \times S$ a tube around the orbit $G.q_0$. Then we get the local description

$$\begin{aligned} (\mathcal{W} //_{\mathcal{O}} G)_{(L)}|U &= T^*S \times_S (\bigsqcup_{q \in U} \mathcal{O} \cap \text{Ann } \mathfrak{g}_q)_{(L)}/G \\ &\cong T^*S \times_S S \times (\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)_{\mathbb{H}}}/H \\ &= T^*S \times (\mathcal{O} //_{\mathcal{O}} H)_{(L_0)_{\mathbb{H}}} \end{aligned}$$

as claimed. The bundle is symplectic by Theorem 4.4.

Assertion (3). The defining property of the reduced symplectic form $\sigma_{(L)}^{\mathcal{O}}$, namely,

$$(\pi_{(L)}^{\mathcal{O}})^* \sigma_{(L)}^{\mathcal{O}} = (\iota_{(L)}^{\mathcal{O}})^* \sigma - (\mu_A | \mathcal{W}_{(L)}^{\mathcal{O}})^* \Omega^{\mathcal{O}}$$

is a well-established fact, see e.g. Bates and Lerman [6, Proposition 11]. Thus it is clear from Proposition 5.1 that

$$\sigma_{(L)}^{\mathcal{O}} = \chi^* \Omega^{Q/G} - \beta_{(L)}^{\mathcal{O}}$$

and it remains to check that $\beta_{(L)}^{\mathcal{O}}$ is a well defined two-form on $(\mathcal{W} //_{\mathcal{O}} G)_{(L)}$. To see this notice firstly that

$$\tilde{\beta} := (\pi_{(L)}^{\mathcal{O}})^* \beta_{(L)}^{\mathcal{O}} = (\iota_{(L)}^{\mathcal{O}})^* dB + (\mu_A|_{\mathcal{W}_{(L)}^{\mathcal{O}}})^* \Omega^{\mathcal{O}} \in \Omega^2(\mathcal{W}_{(L)}^{\mathcal{O}})$$

is G -invariant. Furthermore, we claim that $\tilde{\beta}$ is horizontal, i.e., vanishes upon insertion of a vertical vector field. Indeed, let $(q, \eta, \lambda) \in \mathcal{W}_{(L)}^{\mathcal{O}}$ and consider $\zeta_{Z_1}(q, \eta, \lambda), \xi_2 \in T_{(q, \eta, \lambda)} \mathcal{W}_{(L)}^{\mathcal{O}}$. We proceed locally as in the proof of Proposition 5.1 so that there is a splitting of tangent vectors as $\zeta_{Z_1}(q, \eta, \lambda) = (\zeta_{Z_1}(q), 0, \text{ad}^*(Z_1).\lambda)$ and $\xi_2 = (q'_2, \eta'_2, \text{ad}^*(Y).\lambda)$. Therefore,

$$\begin{aligned} \tilde{\beta}_{(q, \lambda)}(\zeta_{Z_1}(q, \eta, \lambda), \xi_2) &= \langle \text{ad}^*(Z_1).\lambda, Z_2 \rangle - \langle \text{ad}^*(Y).\lambda, Z_1 \rangle + 0 + \langle \lambda, [Z_1, Z_2] \rangle + \langle \lambda, [Z_1, Y] \rangle \\ &= -\langle \lambda, [Z_1, Z_2] \rangle - \langle \lambda, [Z_1, Y] \rangle + \langle \lambda, [Z_1, Z_2] \rangle + \langle \lambda, [Z_1, Y] \rangle = 0. \end{aligned}$$

That is, $\tilde{\beta}$ is a basic form and thus descends to a form $\beta_{(L)}^{\mathcal{O}}$.

Assertion (4) is a pasting together of the results in (2). \square

Corollary 5.7. *Let \mathcal{O} be a coadjoint orbit in the image of the momentum map $\mu_A : \mathcal{W} \rightarrow \mathfrak{g}^*$, and let (L) be in the isotropy lattice of the G -action on \mathcal{W} such that $\mathcal{W}_{(L)}^{\mathcal{O}} := \mu_A^{-1}(\mathcal{O}) \cap \mathcal{W}_{(L)} \neq \emptyset$. Assume further that there is a global slice S such that $Q \cong G/H \times S$. Then we have the global description*

$$\mathcal{W} //_{\mathcal{O}} G = T^*S \times \mathcal{O} //_{\mathcal{O}} H.$$

Moreover, the reduced symplectic form $\sigma_{(L)}^{\mathcal{O}}$ on a symplectic stratum $(\mathcal{W} //_{\mathcal{O}} G)_{(L)} = T^*S \times (\mathcal{O} //_{\mathcal{O}} H)_{(L_0)_{\mathfrak{H}}}$ is given by the formula

$$\sigma_{(L)}^{\mathcal{O}} = \Omega^{Q/G} - \Omega_{(L_0)_{\mathfrak{H}}}^{\mathcal{O}}$$

where $\Omega_{(L_0)_{\mathfrak{H}}}^{\mathcal{O}}$ is the canonically reduced symplectic form on $(\mathcal{O} //_{\mathcal{O}} H)_{(L_0)_{\mathfrak{H}}}$, and L_0 is a subgroup of H which is conjugate to L within G .

Proof. This is an immediate consequence of Theorems 4.4 and 5.6. \square

6. SPIN CALOGERO-MOSER SYSTEMS

In this section we give a mechanical application of Theorem 5.6 to obtain spin Calogero-Moser models. This approach follows, in essence, the idea of Kazhdan, Kostant and Sternberg [17] that such models may be obtained via projection of geodesic systems on Lie groups or Lie algebras. This construction can be carried out in various guises and at different levels of generality. For the simplest case we describe the way Theorem 5.6 allows to understand this projection procedure in the following subsection. The emphasis is here on the use of the mechanical connection.

6.A. The construction based on cotangent bundle reduction. Let G be a (real or complex) simple Lie group, \mathfrak{g} its Lie algebra, \mathfrak{h} a Cartan sub-algebra, and H a corresponding Cartan subgroup. Then we consider either $Q = G_{(H)}$ acted upon by G via conjugation or $Q = \mathfrak{g}_{(H)}$ acted upon by $\text{Ad}(G)$. Note that $G_{(H)}$ is open dense in G and $\mathfrak{g}_{(H)}$ is open dense in \mathfrak{g} . We will see below that choosing $Q = \mathfrak{g}_{(H)}$ leads to Calogero-Moser systems with rational potential while choosing $Q = G_{(H)}$ leads to Calogero-Moser systems with trigonometric potential. As in the

construction of the previous sections we may then consider the lifted G -action on T^*Q which is Hamiltonian with equivariant momentum map $\mu : T^*Q \rightarrow \mathfrak{g}^*$. Since G is assumed simple we can use the Killing form B to identify $T^*Q \cong_B TQ$ and $\mathfrak{g}^* \cong_B \mathfrak{g}$ whence μ becomes a mapping $\mu : TQ \rightarrow \mathfrak{g}$. Let \mathcal{O} be an adjoint orbit in the image of μ . Via the Killing form we may thus define a mechanical connection A on $Q \rightarrow Q/G$ as in Subsection 5.A. By Theorem 5.6 and its corollary the singular symplectic quotient of TQ at \mathcal{O} is therefore given by

$$\mu^{-1}(\mathcal{O})/G = TQ//_{\mathcal{O}}G \cong_A T(Q/G) \times \mathcal{O}//_0H,$$

and this space is called a SPIN CALOGERO-MOSER SPACE. (Recall that $\mathcal{O}//_0H = (\mathcal{O} \cap \mathfrak{h}^\perp)/H$.) This terminology is justified as follows. Use the left multiplication in the group to trivialize the tangent bundle as $TQ = Q \times \mathfrak{g}$. Let $\mathcal{H} : Q \times \mathfrak{g} \rightarrow \mathbb{R}$, $(q, X) \mapsto \frac{1}{2}B(X, X)$ denote the free Hamiltonian. In the notation of Proposition 5.1 the reduced Hamiltonian system on $T(Q/G) \times \mathcal{O}//_0H$ is thus given by Hamiltonian reduction of $(\mathcal{W}, \sigma, \psi^*\mathcal{H})$ at \mathcal{O} . Since the Hamiltonian in this picture is given by

$$(\psi^*\mathcal{H})(q, \eta, \lambda) = \frac{1}{2}B(\eta, \eta) + \frac{1}{2}B(A_q^*(\lambda), A_q^*(\lambda))$$

for $(q, \eta, \lambda) \in \mathcal{W}$ one thus needs to compute $A_q^*(\lambda)$. In fact, it obviously suffices to compute $A_q^*(\lambda)$ for $q \in Q_H$. The crucial point now is that with respect to the identifications $T^*Q = TQ$ and $\mathfrak{g}^* = \mathfrak{g}$ we have

$$A_q^*(\lambda) = \zeta(q)(\check{\mathbb{I}}_q^{-1}(\lambda))$$

where $\check{\mathbb{I}}_q : \mathfrak{g}_q^\perp \rightarrow \mathfrak{g}_q^\perp$ is the G_q -equivariant isomorphism obtained from the locked inertia tensor \mathbb{I} . This object can be computed using structure theory, and we shall do so in the next paragraph.

The rational case. Assume that \mathfrak{g} is a complex simple Lie algebra. (The case of real simple Lie algebras works analogously.) Let $Q = \mathfrak{g}_{(H)}$, let $\Delta \subset \mathfrak{h}^*$ be a root system, Δ^+ a system of positive roots, and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ the corresponding root space decomposition. For each $\alpha \in \Delta$ we choose a vector $E_\alpha \in \mathfrak{g}_\alpha$ such that $B(E_\alpha, E_{-\alpha}) = 1$. Assume that $q \in Q_H = \mathfrak{h}_{\text{reg}}$ where $\mathfrak{h}_{\text{reg}}$ denotes the set of regular elements. Then we may write $\lambda \in \mathcal{O} \cap \mathfrak{h}^\perp$ as $\lambda = \sum_{\alpha \in \Delta} \lambda_\alpha E_\alpha$ and $Z := \check{\mathbb{I}}_q^{-1}(\lambda) \in \mathfrak{h}^\perp$ as $Z = \sum_{\alpha \in \Delta} z_\alpha E_\alpha$. Since $\zeta_Z(q) = \text{ad}(Z).q$ it follows that

$$\begin{aligned} \lambda_\alpha &= B(\lambda, E_{-\alpha}) = \mathbb{I}_q(Z, E_{-\alpha}) = B(\zeta_Z(q), \zeta_{E_{-\alpha}}(q)) = B(Z, -\alpha(q)^2 E_{-\alpha}) \\ &= -z_\alpha \alpha(q)^2. \end{aligned}$$

Therefore, we find that $Z = -\sum_{\alpha \in \Delta} \lambda_\alpha \alpha(q)^{-2} E_\alpha$, and

$$(R) \quad A_q^*(\lambda) = \zeta_Z(q) = +\text{ad}(q) \sum_{\alpha \in \Delta} \frac{\lambda_\alpha}{\alpha(q)^2} E_\alpha = \sum_{\alpha \in \Delta} \frac{\lambda_\alpha}{\alpha(q)} E_\alpha.$$

This in turn implies that the reduced Hamiltonian $(\psi^*\mathcal{H})_0$ is given by

$$\begin{aligned} (\psi^*\mathcal{H})_0(q, \eta, [\lambda]_H) &= \frac{1}{2}B(\eta, \eta) + \frac{1}{2}B\left(\sum_{\alpha \in \Delta} \frac{\lambda_\alpha}{\alpha(q)} E_\alpha, \sum_{\alpha \in \Delta} \frac{\lambda_\alpha}{\alpha(q)} E_\alpha\right) \\ &= \frac{1}{2}B(\eta, \eta) - \sum_{\alpha \in \Delta^+} \frac{\lambda_\alpha \lambda_{-\alpha}}{\alpha(q)^2} \end{aligned}$$

where $(q, \eta, [\lambda]_H) \in T(Q/G) \times \mathcal{O}//_0H = C_{\text{reg}} \times \mathfrak{h} \times \mathcal{O}//_0H$ and where C_{reg} denotes the interior of a Weyl chamber. This function is the Hamiltonian of the rational Calogero-Moser system with spin.

The trigonometric case. Let $Q = G_{(H)}$ and continue the notation regarding the structure theoretic objects of the previous paragraph. Assume $q = \exp a \in Q_H = H_{\text{reg}}$ whence $a \in \mathfrak{h}_{\text{reg}}$. Consider $\lambda \in \mathcal{O} \cap \mathfrak{h}^\perp$ and Z as above. Since $\zeta_Z(q) = \text{Ad}(q^{-1}).Z - Z = (e^{-\text{ad}(a)} - 1).Z$ we may compute

$$\lambda_\alpha = B(\lambda, E_{-\alpha}) = \mathbb{I}_q(Z, E_{-\alpha}) = (e^{-\alpha(a)} - 1)(e^{\alpha(a)} - 1)z_\alpha = (2 - 2 \cosh \alpha(a))z_\alpha$$

and thus obtain the adjoint to the mechanical connection

(T)

$$A_q^*(\lambda) = \zeta_Z(q) = \frac{1}{2} \sum_{\alpha \in \Delta} \frac{2}{e^{-\alpha(a)} - 1} \lambda_\alpha E_\alpha = \frac{1}{2} \sum_{\alpha \in \Delta} \lambda_\alpha E_\alpha + \frac{1}{2} \sum_{\alpha \in \Delta} \coth\left(\frac{-\alpha(a)}{2}\right) \lambda_\alpha E_\alpha$$

in the same way as above. Therefore, the reduced Hamiltonian $(\psi^*\mathcal{H})_0$ is given by

$$(\psi^*\mathcal{H})_0(q, \eta, [\lambda]_H) = \frac{1}{2}B(\eta, \eta) + \frac{1}{2}B(\zeta_Z(q), \zeta_Z(q)) = \frac{1}{2}B(\eta, \eta) - \frac{1}{4} \sum_{\alpha \in \Delta^+} \frac{\lambda_\alpha \lambda_{-\alpha}}{\sinh^2 \alpha(a)}$$

where $(q, \eta) \in T(Q_{(H)}/G) = T(H_{\text{reg}}/W)$ and $[\lambda]_H \in \mathcal{O} //_0 H$, and where $W = N(H)/H$ is the Weyl group. This function is the Hamiltonian of the trigonometric Calogero-Moser system with spin.

Remark 6.1 (Mechanical connection & classical dynamical r -matrix). It is noted in the introduction that the idea of obtaining Calogero-Moser systems through Hamiltonian reduction is originally due to Kazhdan, Kostant and Sternberg [17]. However, a completely different approach to obtain such systems was taken by Li and Xu [19, 20]. They used an analysis based on classical dynamical r -matrices (associated to complex simple Lie algebras) to directly write down the Hamiltonian of spin Calogero-Moser systems (associated to complex simple Lie algebras). (See also Fehér and Pusztai [14, Section 2] for an outline of this construction.) This approach is based on the classification of classical dynamical r -matrices of Etingof and Varchenko [13]. It was noticed by Fehér and Pusztai [14] that one can obtain the same Calogero-Moser models which appear in [19, 20] through Hamiltonian reduction of cotangent bundles. Moreover, constructing certain new spin Calogero-Moser models it was observed by [14, Proof of Prop. 3] that dynamical r -matrices appear in the process of Hamiltonian reduction of T^*G where G is a complex or real simple Lie group acting on itself by twisted conjugation. However, the relationship of the r -matrix and the reduction approach was still mysterious in the sense that there was no explanation for it other than the computations obviously yielding the correct results. We claim that this relationship can be further explained in a geometric framework using the mechanical connection.¹ Indeed, in [13] a classical dynamical r -matrix associated to a complex simple Lie algebra \mathfrak{g} is defined as a meromorphic function $r : \mathfrak{h}^* =_B \mathfrak{h} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ which satisfies the classical dynamical Yang-Baxter equation (CDYBe) and certain other conditions ([13, Subsection 3.2]) in the completed tensor product $\mathfrak{g} \hat{\otimes} \mathfrak{g}$. Via the isomorphism $\mathfrak{g} \otimes \mathfrak{g} \cong \text{hom}(\mathfrak{g}^*, \mathfrak{g}) =_B \text{hom}(\mathfrak{g}, \mathfrak{g})$ we may think of a classical dynamical r -matrix as a meromorphic function $R : \mathfrak{h} \rightarrow \text{hom}(\mathfrak{g}, \mathfrak{g})$ subject to the appropriate equations.

For the *rational case* let us use Equation (R) to define a holomorphic function $R : \mathfrak{h}_{\text{reg}} \rightarrow \text{hom}(\mathfrak{g}, \mathfrak{g})$ by

$$R(q)(\lambda) := A_q^*(\lambda) = \sum_{\alpha \in \Delta} \frac{\lambda_\alpha}{\alpha(q)} E_\alpha$$

By [13, Theorem 3.2] any classical dynamical r -matrix associated to a complex simple Lie algebra \mathfrak{g} with coupling constant $\epsilon = 0$ is of this form (where $X = \Delta$, $C = 0$ and $\nu = 0$ in the notation of [13, Theorem 3.2]).

¹This point of view essentially evolved during discussions with Laszlo Fehér.

For the *trigonometric case* let us use Equation (T) to define a holomorphic function $R : \mathfrak{h}_{\text{reg}} \rightarrow \text{hom}(\mathfrak{g}, \mathfrak{g})$ by $R(a)(\lambda) = \frac{1}{2}\lambda$ for $\lambda \in \mathfrak{h}$, and

$$\begin{aligned} R(a)(\lambda) &:= A_{\text{exp } a}^*(\lambda) = \frac{1}{2} \sum_{\alpha \in \Delta} \lambda_{\alpha} E_{\alpha} + \frac{1}{2} \sum_{\alpha \in \Delta} \coth\left(\frac{-\alpha(a)}{2}\right) \lambda_{\alpha} E_{\alpha} \\ &= i_{\lambda} \left(\frac{1}{2} \Omega + \frac{1}{2} \sum_{\alpha \in \Delta} \coth\left(\frac{\alpha(a)}{2}\right) E_{\alpha} \otimes E_{-\alpha} \right) \end{aligned}$$

for $\lambda \in \mathfrak{h}^{\perp}$. Here $\Omega = \sum_{i=1}^l x_i \check{B}(x_i) + \sum_{\alpha \in \Delta} E_{\alpha} \check{B}(E_{-\alpha})$ is the Casimir element of (\mathfrak{g}, B) where x_1, \dots, x_l is an orthonormal basis of \mathfrak{h} , and $X = \Delta_+$, $C_{i,j} = 0$, $\epsilon = 1$ and $\nu = 0$ in the notation of [13, Theorem 3.10]. By [13, Theorem 3.10] any classical dynamical r -matrix associated to a simple Lie algebra with coupling constant $\epsilon = 1$ is of this form.

We view this as a new geometric explanation of why it is possible to associate Calogero-Moser systems to classical dynamical r -matrices. It remains a goal for future work to find a general relationship between the condition (i.e., CDYBe) defining classical dynamical r -matrices and the properties (such as $\text{Curv}^A = 0$) of the mechanical connection.

6.B. $\text{SL}(m, \mathbb{C})$ by hand. As an example consider $G = \text{SL}(m, \mathbb{C})$. Here we work along the lines of Kazhdan, Kostant, Sternberg [17, Section 2] who considered the case $G = \text{SU}(m, \mathbb{C})$. See also Alekseevsky, Kriegl, Losik, Michor [2, Section 5.7]. The point to this example is that we try to say as much as possible about the reduced phase space by using an *ad hoc* approach.

Let $\mathcal{O} = \text{Ad}(G)Z_0$ be an orbit passing through a semi-simple element Z_0 . Consider $(a, \alpha) \in G_r \times \mathfrak{g}$ with $\alpha - a\alpha a^{-1} = \mu(a, \alpha) = Z$. Note that $\mu : G_r \times \mathfrak{g} \rightarrow \mathfrak{g}$ is the equivariant momentum map of the action which is obtained by lifting the G -action on G_r by conjugation to the (co-)tangent bundle $G_r \times \mathfrak{g}$. As usual G_r denotes the set of regular elements, that is, G_r consists of those matrices that have m different eigenvalues. Moreover, we let H denote the subgroup of diagonal matrices, and $H_r := H \cap G_r$. Via the $\text{Ad}(G)$ -action we can bring a in diagonal form with entries $a_i \neq a_j$ for $i \neq j$. Since $Z_{ij} = \alpha_{ij} - \frac{a_i}{a_j} \alpha_{ij}$ the following are coordinates on $(\mu^{-1}(\mathcal{O}) \cap (G_r \times \mathfrak{g})) / \text{Ad}(G) = T^*G_r //_{\mathcal{O}} G$:

- a_i for $i = 1, \dots, m$.
- $\alpha_i := \alpha_{ii}$ for $i = 1, \dots, m$.
- $\alpha_{ij} = (1 - \frac{a_i}{a_j})^{-1} Z_{ij}$ for $i \neq j$.

These coordinates give an identification

$$(\mu^{-1}(\mathcal{O}) \cap (G_r \times \mathfrak{g})) / \text{Ad}(G) = (T^*H_r \times (\mathcal{O} \cap \mathfrak{h}^{\perp}) / \text{Ad}(H)) / W$$

where $W = N(H)/H$ is the Weyl group. CLAIM: *If \mathcal{O} is an orbit which is of minimal non-zero dimension then we have that $\mathcal{O} \cap \mathfrak{h}^{\perp} / \text{Ad}(H) = \{\text{point}\}$. Moreover, the reduced phase space can be described as $(\mu^{-1}(\mathcal{O}) \cap (G_d \times \mathfrak{g})) / \text{Ad}(G) \cong T^*H_r / W$.*

Here G_d denotes the open and dense subset of all diagonalizable elements in $\text{SL}(m, \mathbb{C})$. Indeed, let $\mu(a, \alpha) = Z \in \mathcal{O} \cap \mathfrak{h}^{\perp}$ with a in diagonal form. Thus $Z = vw^t - cI$ where $c := \frac{1}{m} \langle v, w \rangle \neq 0$, $v, w \in \mathbb{C}^m$, and w^t is the transposed to the column vector w . Since $Z \in \mathfrak{h}^{\perp}$ we infer that $v_i w_i = c$. Hence

$$\mathcal{O} \cap \mathfrak{h}^{\perp} = \left\{ \left(\frac{c}{v_1} v, \dots, \frac{c}{v_m} v \right) - cI : v_i \in \mathbb{C} \setminus \{0\} \right\}.$$

Take such an $(\frac{c}{v_1} v, \dots, \frac{c}{v_m} v) - cI =: Z_1$. Let $h = \prod_{i=1}^m v_i \cdot \text{diag}(v_1^{-1}, \dots, v_m^{-1})$. Then we can bring Z_1 into the normal form $\text{Ad}(h)Z_1 = c(1)_{ij} - cI$ where $(1)_{ij}$ denotes the $m \times m$ -matrix with all entries equal to 1. Finally note that $\alpha_{ij} - \frac{a_i}{a_j} \alpha_{ij} = \frac{c}{v_j} v_i \neq 0$ implies that $a = \text{diag}(a_1, \dots, a_m)$ is actually regular.

The coordinates for $T^*G_r//_{\mathcal{O}}G$ found above by evaluating the momentum constraint equation and factoring out the G -action have been the motivating point for the formulation of the general Theorem 5.6.

6.C. Application: Hermitian matrices. Consider V the space of complex Hermitian $n \times n$ matrices as the configuration space to start from. This space shall be endowed with the inner product $V \times V \rightarrow \mathbb{R}$, $(a, b) \mapsto \text{Tr}(ab)$. Moreover, we let $G = \text{SU}(n, \mathbb{C})$ act on V by conjugation. Clearly this action leaves the trace form invariant. Via the inner product we can trivialize the cotangent bundle as $T^*V = V \times V^* = V \times V$, and the cotangent lifted action of G is simply given by the diagonal action. The canonical symplectic form on T^*V is given by

$$\Omega_{(a, \alpha)}((a_1, \alpha_1), (a_2, \alpha_2)) = \text{Tr}(\alpha_2 a_1) - \text{Tr}(\alpha_1 a_2).$$

The free Hamiltonian on $T^*V = V \times V$ is given by

$$H_{\text{free}} : (a, \alpha) \mapsto \frac{1}{2} \text{Tr}(\alpha a).$$

Trajectories of this Hamiltonian are given by straight lines of the form $t \mapsto a + t\alpha$ in the configuration space V .

Let us further identify $\mathfrak{su}(n)^* = \mathfrak{su}(n)$ via the Killing form. The momentum mapping is then given by

$$\mu : (a, \alpha) \mapsto [a, \alpha] = \text{ad}(a).\alpha.$$

Consider also an orbit \mathcal{O} together with its canonically induced symplectic structure in the image of the momentum mapping.

Assumption: The orbit \mathcal{O} is such that $\mu^{-1}(\mathcal{O}) \subseteq V_r \times V$. Here V_r denotes the set of regular elements in V with respect to the G action.

This assumption is, for example, fulfilled if the projection from \mathcal{O} to any root space is non-trivial. On the other hand, if the assumption is not satisfied for a particular orbit \mathcal{O} one can also consider the restricted G -action on V_r and proceed with reduction of the Hamiltonian system $(T^*V_r, \Omega, H_{\text{free}})$ at the orbit level \mathcal{O} . This has, however, the disadvantage that the Hamiltonian flow lines may leave $T^*V_r//_{\mathcal{O}}G$ in finite time.

Let Σ denote the subspace of V consisting of diagonal matrices. Then Σ is a section of the G -action on V , see Section 7. Further, we define $\Sigma_r := V_r \cap \Sigma$. Within Σ we choose the positive Weyl chamber $C := \{\text{diag}(q_1, \dots, q_n) : q_1 > \dots > q_n\}$ so that $C = \Sigma/W = V/G$ where $W = W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$. Thus $C_r := \Sigma_r \cap C$ may be considered as a global slice for the G -action on V_r so that $G/M \times C_r \cong V_r$, $(gM, a) \mapsto g.a$ where $M := Z_G(\Sigma_r) = Z_G(\Sigma)$. That is, M is the subgroup of $\text{SU}(n)$ consisting of diagonal matrices only. Now we may apply Corollary 5.7 to get

$$T^*V//_{\mathcal{O}}G = T^*C_r \times \mathcal{O}//_0M$$

as symplectic stratified spaces. The strata are of the form

$$(T^*V//_{\mathcal{O}}G)_{(L)} = T^*C_r \times (\mathcal{O}//_0M)_{(L_0)_{\bar{M}}^G}$$

where L_0 is a subgroup of M conjugate to L within G . Moreover, the reduced symplectic structure $\sigma_{(L)}^{\mathcal{O}}$ on $(T^*V//_{\mathcal{O}}G)_{(L)}$ is of product form, i.e.,

$$\sigma_{(L)}^{\mathcal{O}} = \Omega^{C_r} - \Omega_{(L_0)_{\bar{M}}^G}^{\mathcal{O}}$$

where $\Omega_{(L_0)_{\bar{M}}^G}^{\mathcal{O}}$ is the canonically reduced symplectic form on $(\mathcal{O}//_0M)_{(L_0)_{\bar{M}}^G}$.

From the general theory (Theorem 3.1) we know that the Hamiltonian H_{free} reduces to a Hamiltonian $H_{\text{CM}}^{(L)}$ on the stratum $(T^*V//_{\mathcal{O}}G)_{(L)}$, and that integral curves of

H_{free} project to integral curves of $H_{\text{CM}}^{(L)}$. In particular, the dynamics remain confined to the symplectic stratum. The reduced Hamiltonian is thus given by

$$H_{\text{CM}}^{(L)}(q, p, [\lambda]) = H_{\text{free}}(q, p + A_q^*(\lambda))$$

where $[\lambda]$ is the class of λ in $(\mathcal{O} //_0 M)_{(L_0)_M^G}$ and $A_q^* : \mathfrak{g}_q^\perp = \mathfrak{m}^\perp \rightarrow T_q(G \cdot q) = \Sigma^\perp$ is the point wise dual to the mechanical connection as introduced in Section 5. Assume that $q = \text{diag}(q_1 > \dots > q_n)$ and that $\lambda = (\lambda_{ij})_{ij} \in (\mathcal{O} \cap \mathfrak{m}^\perp)_{(L_0)_M^G}$. Then

$$A_q^*(\lambda)_{ij} = \frac{\lambda_{ij}}{q_i - q_j} \text{ for } i \neq j, \text{ and } A_q^*(\lambda)_{ii} = 0.$$

Therefore, for $p = \text{diag}(p_1, \dots, p_n) \in \Sigma$ and $q, [\lambda]$ as introduced we obtain

$$\begin{aligned} H_{\text{CM}}^{(L)}(q, p, [\lambda]) &= \frac{1}{2} \text{Tr}(p)^2 + \frac{1}{2} \text{Tr}(A_q^*(\lambda))^2 = \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{\lambda_{ij} \lambda_{ji}}{(q_i - q_j)(q_j - q_i)} \\ &= \frac{1}{2} \sum_{i=1}^n p_i^2 + \sum_{i > j} \frac{|\lambda_{ij}|^2}{(q_i - q_j)^2} \end{aligned}$$

since $\lambda_{ji} = -\overline{\lambda_{ij}}$ and $\text{Tr}(p A_q^*(\lambda)) = \text{Tr}(A_q^*(\lambda) p) = 0$. This is the Hamiltonian function of the Calogero-Moser system with spin. Integrability of this system in the non-commutative sense is proved in the next section in a more general context.

6.D. Application: Polar representations of compact Lie groups. The idea of considering polar representations of compact Lie groups to obtain new versions of Spin Calogero-Moser systems is due to Alekseevsky, Kriegl, Losik, Michor [2].

As in Section 7 let V be a real Euclidean vector space and G a connected compact Lie group that acts on V via a polar representation. Via the inner product we consider the cotangent bundle of V as a product $T^*V = V \times V$. The canonical symplectic form Ω is thus given by

$$\Omega_{(a, \alpha)}((a_1, \alpha_1), (a_2, \alpha_2)) = \langle \alpha_2, a_1 \rangle - \langle \alpha_1, a_2 \rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product on V . The free Hamiltonian on $T^*V = V \times V$ is given by

$$H_{\text{free}} : (a, \alpha) \mapsto \frac{1}{2} \langle \alpha, \alpha \rangle.$$

Trajectories of this Hamiltonian are given by straight lines of the form $t \mapsto a + t\alpha$ in the configuration space V .

Of course, the cotangent lifted action of G is just the diagonal action of G on $V \times V$. By Section 7 we may think of the action by G on V as a symmetric space representation and thus consider $\mathfrak{g} \oplus V =: \mathfrak{l}$ as a real semi-simple Lie algebra with Cartan decomposition into \mathfrak{g} and V , and with bracket relations $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$, $[\mathfrak{g}, V] \subseteq V$, and $[V, V] \subseteq V$. The momentum mapping corresponding to the G -action on $T^*V = V \times V$ is now given by

$$\mu : V \times V \longrightarrow \mathfrak{g}^* = \mathfrak{g}, \quad (a, \alpha) \mapsto [a, \alpha] = \text{ad}(a) \cdot \alpha$$

where we identify $\mathfrak{g} = \mathfrak{g}^*$ via an $\text{Ad}(G)$ -invariant inner product. Consider also an orbit \mathcal{O} together with its canonically induced symplectic structure in the image of the momentum mapping.

Assumption: The orbit \mathcal{O} is such that $\mu^{-1}(\mathcal{O}) \subseteq V_r \times V$. Here V_r denotes the set of regular elements in V with respect to the G action.

We proceed as above, and let Σ denote a fixed section of the G -action on V , consider C a Weyl chamber in Σ , and put $M := Z_G(\Sigma)$. We may apply Corollary 5.7 to get

$$T^*V //_{\mathcal{O}} G = T^*C_r \times \mathcal{O} //_0 M$$

as symplectic stratified spaces. The strata are of the form

$$(T^*V//_{\mathcal{O}G})_{(L)} = T^*C_r \times (\mathcal{O}//_0M)_{(L_0)M}^{\mathcal{G}}$$

where L_0 is a subgroup of M conjugate to L within G . Moreover, the reduced symplectic structure $\sigma_{(L)}^{\mathcal{O}}$ on $(T^*V//_{\mathcal{O}G})_{(L)}$ is of product form, i.e.,

$$\sigma_{(L)}^{\mathcal{O}} = \Omega^{C_r} - \Omega_{(L_0)M}^{\mathcal{O}}$$

where $\Omega_{(L_0)M}^{\mathcal{O}}$ is the canonically reduced symplectic form on $(\mathcal{O}//_0M)_{(L_0)M}^{\mathcal{G}}$.

From the general theory we know that the Hamiltonian H_{free} reduces to a Hamiltonian $H_{\text{CM}}^{(L)}$ on the stratum $(T^*V//_{\mathcal{O}G})_{(L)}$, and that integral curves of H_{free} project to integral curves of $H_{\text{CM}}^{(L)}$. In particular the dynamics remain confined to the symplectic stratum. The reduced Hamiltonian is thus given by

$$H_{\text{CM}}^{(L)}(q, p, [Z]) = H_{\text{free}}(q, p + A_q^*(\lambda))$$

where $[Z]$ is the class of Z in $(\mathcal{O}//_0M)_{(L_0)M}^{\mathcal{G}}$ and $A_q^* : \mathfrak{g}_q^\perp = \mathfrak{m}^\perp \rightarrow T_q(G \cdot q) = \Sigma^\perp$ is the point wise dual to the mechanical connection as introduced in Section 5. Let $q \in C_r$, $p = \sum_{i=1}^l p_i B_0^i$, and $Z = \sum_{\lambda \in R} \sum_{i=1}^{k_\lambda} z_\lambda^i E_\lambda^i \in (\mathcal{O} \cap \mathfrak{m}^\perp)_{(L_0)M}^{\mathcal{G}}$ where $l = \dim \Sigma$ and $k_\lambda = \frac{1}{2} \dim \mathfrak{l}_\lambda$. The notation here is as in Section 7, and $R = R(\mathfrak{l}, \Sigma) \subseteq \Sigma^*$ denotes the set of restricted roots, in particular. With these definitions the dual mapping to the mechanical connection is given by

$$A_q^*(Z) = \sum_{\lambda \in R} \sum_{i=1}^{k_\lambda} \frac{z_\lambda^i}{\lambda(q)} B_\lambda^i.$$

Note that $\lambda(q) \neq 0$ for all $\lambda \in R$ since $q \in C_r$ is regular. The reduced Hamiltonian thus computes to

$$H_{\text{CM}}^{(L)}(q, p, [Z]) = \frac{1}{2} \langle p + A_q^*(Z), p + A_q^*(Z) \rangle = \frac{1}{2} \sum_{i=1}^l p_i^2 + \frac{1}{2} \sum_{\lambda \in R} \frac{\sum_{i=1}^{k_\lambda} z_\lambda^i z_\lambda^i}{\lambda(q)^2}.$$

The reduced Hamiltonian system $(T^*V//_{\mathcal{O}G}, \sigma^{\mathcal{O}}, H_{\text{CM}})$ is thus a new version of a Calogero-Moser system with spin. It is, in fact, a stratified Hamiltonian system in the sense that it is a Hamiltonian system on each symplectic stratum $(T^*V//_{\mathcal{O}G})_{(L)}$, and the dynamics remain confined to these strata.

We now show that the thus obtained Calogero-Moser system is integrable in the non-commutative sense. To do so we will use Theorem 6.2. We start by choosing coordinates $q_1, \dots, q_n, p_1, \dots, p_n$ on $T^*V = V \times V$ such that the Poisson bracket of functions $f, g \in C^\infty(V \times V)$ is given by the usual equation $\{f, g\} = \sum_{i=1}^n (\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i})$. Moreover we assume that $q_1, \dots, q_l, p_1, \dots, p_l$ are coordinates on $\Sigma \times \Sigma \hookrightarrow V \times V$. Let us now consider the map

$$\Phi : V \times V \longrightarrow \Sigma^\perp \times V$$

given by projection, and endow $\Sigma^\perp \times V$ with the inherited Poisson structure. Clearly, $C^\infty(\Sigma^\perp \times V)$ has a center and this is just generated by p_1, \dots, p_l . Thus we may identify $Z(C^\infty(\Sigma^\perp \times V)) = C^\infty(\Sigma)$. Now the set of all first integrals of H_{free} , i.e.,

$$\mathcal{F}_{H_{\text{free}}} = \{F \in C^\infty(V \times V) : \{F, H\} = 0\}$$

can be identified with $C^\infty(\Sigma^\perp \times V)$ via Φ since H_{free} factors over the projection onto the second factor and is G -invariant, and thus can be considered as a function on Σ . Therefore,

$$\dim V \times V = \dim \Sigma^\perp \times V + \dim \Sigma = \text{ddim } \mathcal{F}_{H_{\text{free}}} + \text{ddim } Z(C^\infty(\Sigma^\perp \times V)),$$

and we are exactly in the situation of the following theorem to conclude non-commutative integrability of the reduced system.

Theorem 6.2. *Assume the Hamiltonian system (M, ω, H) is invariant under a Hamiltonian action of a compact Lie group G . If (M, ω, H) is non-commutatively integrable (Definition 6.3) then the reduced system is integrable as well:*

- *The singular Poisson reduced system is non-commutatively integrable.*
- *The singular symplectic reduced system is non-commutatively integrable.*

Proof. This theorem is proved by Zung [50, Theorem 2.3]. For material on singular reduction we refer to Ortega and Ratiu [30] and Section 3. \square

The idea of non-commutative integrability under the name of degenerate integrability is due to Nehorošev [28] who also introduced the appropriate concept of action-angle variables. This section follows mainly the approach of Zung [50, 51]. See also Mishchenko and Fomenko [26]. The following definition is less general than that given in the above cited references but better suited for the applications in this paper.

Definition 6.3. Let $(M, \{ \cdot, \cdot \})$ be a Poisson manifold, and consider a Hamiltonian function $H : M \rightarrow \mathbb{R}$. We denote the Poisson sub-algebra of all first integrals of H by \mathcal{F}_H , that is

$$\mathcal{F}_H := \{F \in C^\infty(M) : \{F, H\} = 0\}.$$

The Hamiltonian system is called NON-COMMUTATIVELY INTEGRABLE if there is a finite dimensional Poisson vector space W and a generalized momentum map $\Phi : M \rightarrow W$ which is a Poisson morphism with respect to the Poisson structure on W such that the following are satisfied.

- $\Phi^* : C^\infty(W) \rightarrow \mathcal{F}_H$ is an isomorphism of Lie-Poisson algebras.
- $\dim M = \text{ddim } C^\infty(W) + \text{ddim } Z(C^\infty(W))$ where $Z(C^\infty(W))$ denotes the commutative sub-algebra of Casimir functions on W , and $\text{ddim } C^\infty(W) = \dim W$ is the functional dimension of $C^\infty(W)$.

It is crucial in the formulation of the above theorem that $\dim M = \text{ddim } \mathcal{F}_H + \text{ddim } Z(\mathcal{F}_H)$, and \mathcal{F}_H is the set of *all* first integrals of H .

7. APPENDIX: POLAR REPRESENTATIONS

Let V be a real Euclidean vector space, and G be a connected compact Lie group. Further, let $\rho : G \rightarrow \text{SO}(V, \langle \cdot, \cdot \rangle)$ be a POLAR REPRESENTATION of G on V . That is, there is subspace $\Sigma \subseteq V$ (a SECTION) such that Σ meets all G -orbits, and does so orthogonally. The following is due to Dadok [9] and is a consequence of his classification of polar actions.

Proposition 7.1. *There exists a connected Lie group \tilde{G} together with a representation $\tilde{\rho} : \tilde{G} \rightarrow \text{SO}(V)$ such that the following hold. There is a real semi-simple Lie algebra \mathfrak{l} with a Cartan decomposition $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{p}$. Moreover, there is a Lie algebra isomorphism $A : \text{Lie}(\tilde{G}) = \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ and a linear isomorphism $B : V \rightarrow \mathfrak{p}$ such that $B(\tilde{\rho}'(X).v) = [A(X), B(v)]$ for all $X \in \tilde{\mathfrak{g}}$ and $v \in V$. Finally, the G -orbits coincide with the \tilde{G} -orbits, that is $V/G = V/\tilde{G}$.*

Proof. See Dadok [9, Proposition 6]. \square

Thus, for the purpose of this paper, it suffices to assume that the representation of G on V is a symmetric space representation whence $\mathfrak{l} = \mathfrak{g} \oplus V$ is a Cartan decomposition, and hence $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$, $[\mathfrak{g}, V] \subseteq V$, and $[V, V] \subseteq \mathfrak{g}$. Therefore, $G \times$

$V \cong L$, $(g, v) \mapsto g \exp(v)$ is a global Cartan decomposition with compact G where $\text{Lie}(L) = \mathfrak{l}$.

An element $v \in V$ is said to be REGULAR (with respect to the G -action) if the orbit $\mathcal{O}(v) = \rho(G).v = G.v$ is of maximal possible dimension. The set of regular elements will be denoted by V_r . The following assertions which are easy to verify are used in Subsection 6.D. (See also Knapp [18, Chapter VI].)

Let $v \in V$. Then, by reason of dimension, $\text{ad}(v)|_{Z_{\mathfrak{g}}(v)^\perp} : Z_{\mathfrak{g}}(v)^\perp \rightarrow Z_V(v)^\perp$ and $\text{ad}(v)|_{Z_V(v)^\perp} : Z_V(v)^\perp \rightarrow Z_{\mathfrak{g}}(v)^\perp$ both are linear isomorphisms.

The set V_r of regular elements is open dense in V . Moreover, $v \in V_r$ if and only if $Z_V(v) =: \Sigma$ is a section in V . This is the case if and only if Σ is maximally Abelian. Let $\Sigma \subset V$ be a section, and put $\mathfrak{m} := Z_{\mathfrak{g}}(\Sigma)$. The set $R = R(\mathfrak{l}, \Sigma) \subseteq \Sigma^*$ shall denote the set of restricted roots. This gives rise to the restricted root space decomposition

$$\mathfrak{l} = \mathfrak{m} \oplus \Sigma \oplus \bigoplus_{\lambda \in R} \mathfrak{l}_\lambda.$$

Any Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{l}$ of \mathfrak{l} is of the form $\mathfrak{h} = \mathfrak{t} \oplus \Sigma$ where $\mathfrak{t} \subseteq \mathfrak{m}$ is a Cartan subalgebra (Lie algebra to a maximal torus) of \mathfrak{g} .

Each restricted root space \mathfrak{l}_λ has an orthonormal basis $E_\lambda^i \in \mathfrak{g}, B_\lambda^i \in V$ where $i = 1, \dots, k_\lambda = \frac{1}{2} \dim \mathfrak{l}_\lambda$, and which is such that $\text{ad}(v)E_\lambda^i = \lambda(v)B_\lambda^i$ and $\text{ad}(v)B_\lambda^i = \lambda(v)E_\lambda^i$ for all $v \in \Sigma$. The vectors E_0^i , where $i = 1, \dots, \dim \mathfrak{m}$, and B_0^j , where $j = 1, \dots, \dim \Sigma$ will denote an orthonormal basis of \mathfrak{m} and Σ respectively.

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