Chaplygin systems associated to semi-simple Lie groups and cases of Hamiltonization

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G-Chaplygin systems

A *non-holonomic system* is a triple $(Q, L, \mathcal{D})$ where

- Q is an n-dimensional configuration manifold;
- $\mathcal{D} \subset TQ$ is a smooth non-integrable distribution of constant rank;
- $L = E_{\text{kin}} - E_{\text{pot}} : TQ \to \mathbb{R}$ is the Lagrangian, and we assume that E_{kin} defines a Riemannian metric μ on Q and $E_{\text{pot}} = V(q)$.

The equations of motion for a curve $q(t)$ in Q, such that $q' \in \mathcal{D}$, are determined by the Lagrange-d'Alembert principle.

A *G-Chaplygin system* is a non-holonomic system $(Q, L, \mathcal{D})$ acted upon by a Lie group G in a free and proper fashion such that $\mathcal{D}$ defines a connection on the principal bundle

$$G \hookrightarrow Q \twoheadrightarrow Q/G =: S.$$

(Also called the principal or purely kinematical case.) In particular, G acts by isometries on (Q, μ).

We do not require that $\mathcal{D}$ is the mechanical connection.

The almost Hamiltonian formulation

Via the metric μ we can associate a Hamiltonian $\mathcal{H}$ to the Lagrangian,

$$\mathcal{H}(q,p) = \frac{1}{2}\mu_q(p,p) + V(q).$$

Let $\mathcal{A}^a \in \Omega^1(Q)$, $a = 1, \dots, \dim G$ be the components of the connection form such that $\mathcal{D} = \ker(\mathcal{A}^a)_a$. In terms of local coordinates (q^i, p_i) on Q the equations of motion are given by

$$X^{\mathcal{M}}(q,p) := \begin{pmatrix} q^i \\ p_i \end{pmatrix}' = \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial p_i} \\ -\frac{\partial \mathcal{H}}{\partial q^i} - \sum_a \lambda_a \mathcal{A}^a(\frac{\partial}{\partial q^i}) \end{pmatrix}$$

where the λ_a are the Lagrange multipliers determined from the supplementary equations $\mu(\mathcal{A}^a, p) = \mathcal{A}^a(v) = 0$. Thus $X^{\mathcal{M}}$ is tangent to

$$\mathcal{M} := \check{\mu}(\mathcal{D}).$$

Intrinsically this writes as

$$i(X^{\mathcal{M}})\Omega^Q = d\mathcal{H} + \sum_a \lambda_a \tau^* \mathcal{A}^a$$

where $\tau : T^*Q \to Q$.

The Bates-Śniatycki construction

Let $\iota : \mathcal{M} \hookrightarrow T^*Q$ and $\mathcal{A} := (\mathcal{A}^a)_a : TQ \to \mathfrak{g}$. The G-action induces an action on $\mathcal{M}$ and there is an induced connection $\iota^*\tau^*\mathcal{A}$ on the principal fiber bundle

$$G \hookrightarrow \mathcal{M} \twoheadrightarrow \mathcal{M}/G = T^*S.$$

Let

$$\mathcal{C} := (\iota^*\tau^*\mathcal{A})^{-1}(0) \subset T\mathcal{M}$$

be the horizontal space. Define $\Omega^{\mathcal{C}}$ to be the fiber-wise restriction to $\mathcal{C}$ of $\iota^*\Omega^Q$, and $(d\mathcal{H})^{\mathcal{C}}$ the restriction of $d\mathcal{H}$ to $\mathcal{C}$. Then

$$i(X^{\mathcal{M}})\Omega^{\mathcal{C}} = (d\mathcal{H})^{\mathcal{C}}. \qquad (1)$$

Theorem ([BS93])

$\Omega^{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ *is non-degenerate.*

Thus we can describe the dynamics in terms of the triple $(\mathcal{M}, \Omega^{\mathcal{C}}, \iota^*\mathcal{H})$ together with (1); $d\Omega^{\mathcal{C}} \neq 0$ in general.

Can we reproduce this structure on T^*S?

Beside: The mechanical case

What happens when $\mathcal{A} : TQ \to \mathfrak{g}$ is the mechanical connection?

Then $\mathcal{M} = J^{-1}(0)$ where $J : T^*Q \to \mathfrak{g}^*$ is the standard momentum map of the lifted G-action. Further, $\mathcal{C}$ is the associated horizontal space of $\mathcal{M} \twoheadrightarrow \mathcal{M}/G = T^*S$, and non-degeneracy of $\Omega^{\mathcal{C}}$ follows from the momentum map equation

$$i(\zeta_X)\Omega^Q = d\langle J, X\rangle \text{ for all } X \in \mathfrak{g}$$

which implies that $\mathrm{Ver}(\mathcal{M} \twoheadrightarrow \mathcal{M}/G) = \ker \iota^*\Omega^Q$.

Therefore, when $\mathcal{A}$ is mechanical,

$$X^{\mathcal{M}} = X_{\mathcal{H}} = \check{\Omega}^{-1}(d\mathcal{H}) \in \mathcal{C} \subset TT^*Q|\mathcal{M}$$

since $X_{\mathcal{H}}$ is tangent to $\mathcal{M}$ by Noether's Theorem.

When $\mathcal{A}$ is *not* the mechanical connection, $\mathcal{M} \neq J^{-1}(0)$, and the dynamics of $X^{\mathcal{M}} \in \mathfrak{X}(\mathcal{M})$ will differ from those of $X_{\mathcal{H}} \in \mathfrak{X}(J^{-1}(0))$.

Non-holonomic reduction: Compression

Let $\rho : \mathcal{M} \twoheadrightarrow \mathcal{M}/G = T^*S$ denote the projection. This map has a fiber-wise inverse:

$$\mathsf{hl}^{\mathcal{A}} : T^*S \cong_{\mu_0} TS \xrightarrow{\mathsf{hl}^{\mathcal{A}}} \mathcal{D} \cong_{\mu} \mathcal{M}$$

where μ_0 is the induced metric on S and $\mathsf{hl}^{\mathcal{A}}$ is the horizontal lift.

Theorem ([BS93, K92])

- $\Omega^{\mathcal{C}}$ *descends to a non-degenerate two-form* Ω_{nh} *on* T^*S.
- $\Omega_{\mathsf{nh}} = \Omega^S - \langle J_G \circ \mathsf{hl}^{\mathcal{A}}, \mathsf{Curv}_0^{\mathcal{A}} \rangle$ *where* $\mathsf{Curv}_0^{\mathcal{A}}$ *is the induced curvature form on* S.
- $\iota^*\mathcal{H}$ *drops to a function* $\mathcal{H}_{\mathsf{c}}$ *on* T^*S.
- *The vector field* $X^{\mathcal{M}}$ *is* ρ*-related to the vector field* $X_{\mathsf{nh}} := (\check{\Omega}_{\mathsf{nh}})^{-1} d\mathcal{H}_{\mathsf{c}}$.

The form Ω_{nh} is, in general, not closed. The non-holonomic system $(Q, L, \mathcal{D})$ is thus encoded in the almost Hamiltonian system $(T^*S, \Omega_{\mathsf{nh}}, \mathcal{H}_{\mathsf{c}})$.

Remarks on Hamiltonization

An *almost Hamiltonian system* is a triple $(M, \Omega, \mathcal{H})$ with Ω non-degenerate. The dynamics are given by $X = \Omega^{-1} d\mathcal{H}$. The system is *Hamiltonizable* if there is a function $F : M \to \mathbb{R}_{>0}$ such that $d(F\Omega) = 0$. Then $(M, F\Omega, \mathcal{H})$ is Hamiltonian with dynamics $F^{-1}X$. If this is the case, $(M, \Omega, \mathcal{H})$ possesses a preserved measure:

$$L_X(F^{d-1}\Omega^d) = L_{F^{-1}X}(F^d\Omega^d) = 0 \quad (2d = \dim M)$$

and F^{d-1} is called the preserved density.

When a preserved measure exists then a Liouville type theorem holds. There are G-Chaplygin systems which do not have a preserved measure.

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Further, associated to (M, Ω) there is a codifferential operator δ:

$$* : \Omega^k(M) \longrightarrow \Omega^{2d-k}(M) \qquad * \psi = \tfrac{1}{d!} i(\Omega^{-1}\psi)\Omega^d$$

where $2d = \dim M$. Now

$$\delta := *d* : \Omega^k(M) \longrightarrow \Omega^{k-1}(M).$$

(M, Ω) is locally conformally symplectic if F exists locally.

Theorem (Libermann 1953)

(1) *If* $\dim M = 4$ *then* (M, Ω) *is l.c.s. iff* $\delta\Omega$ *is closed.*

(2) *If* $\dim M > 4$ *then* (M, Ω) *is l.c.s. iff* $(d-1)d\Omega = \delta\Omega \wedge \Omega$.

From now on $M = T^*B$. Using that $d\Omega = \delta\Omega \wedge \Omega$ for $\dim B = 2$:

Theorem (Chaplygin's multiplier theorem ∼1900)

Let $\dim B = d = 2$. *Let* $(T^*B, \Omega, \mathcal{H})$ *a.H. with* $\mathcal{H}$ *of kinetic energy type such that*

(1) $\Omega = \Omega^{T^*B} + \varphi(q)dq^1 \wedge dq^2 + I(q,p)dq^1 \wedge dq^2$ *with* I *linear in* p. *('Canonical + Magnetic + Semi-basic')*

(2) *There is a preserved density* $F : B \to \mathbb{R}_{>0}$.

Then

$$\delta\Omega = -d \log F \text{ whence } d(F\Omega) = 0.$$

A multiplier theorem for higher dimensions

Theorem

Let $\dim B = d > 2$. *Let* $(T^*B, \Omega, \mathcal{H})$ *a.H. with* $\mathcal{H}$ *of kinetic energy type such that*

(1) $\Omega = \Omega^{T^*B} + \Xi + \Lambda$ *with* Ξ *magnetic, and* Λ *semi-basic and linear in the fibers.*

(2) *There is a preserved density* $F^{d-1} : B \to \mathbb{R}_{>0}$.

(3) $(d-1)d\Omega = \delta\Omega \wedge \Omega$

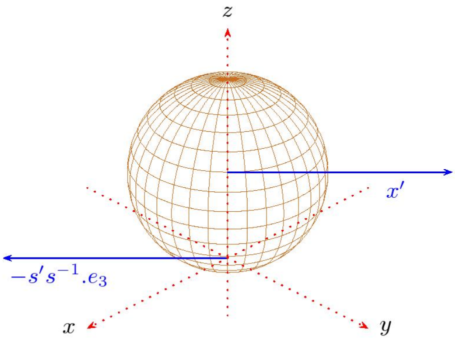
Then

$$(d-1)\delta\Omega = -d \log F \text{ whence } d(F\Omega) = 0.$$

This theorem holds under similar assumptions when internal variables are adjoined in the form of a minimal coupling procedure ('Weinstein space construction') involving $(T^*B, \Omega, \mathcal{H})$ and a symplectic space (F, Ω^F).

Part II: Chaplygin's rolling ball

The Chaplygin ball is a ball whose center of mass lies at the geometric center and which rolls without slipping on a horizontal table. Mass distribution may be inhomogeneous.



Constraints

The horizontal component of the contact point's angular velocity equals minus the ball's linear velocity. Or: The contact point has zero total velocity.

The n-dimensional Chaplygin ball

The configuration space of the n-dimensional Chaplygin ball is

$$Q := S \times V := \mathrm{SO}(n) \times \mathbb{R}^{n-1}.$$

A curve $(s(t), x(t))$ is an allowed motion iff

$$(s' s^{-1}, x') \in \widetilde{\mathcal{D}} := \{(\widetilde{u}, x') \in \mathfrak{so}(n)_R \times V : \widetilde{u}.e_n = (x', 0)^t\}.$$

Here $\mathfrak{so}(n)_R$ is the Lie algebra of *right* invariant vector fields on S. In terms of the *left* trivialization, $TS = S \times \mathfrak{so}(n)$, the Hamiltonian reads

$$\mathcal{H} = \frac{1}{2}\langle \mathbb{I}u, u\rangle + \frac{1}{2}\langle x', x'\rangle_V$$

where $u = s^{-1}s'$, $\langle ., . \rangle$ is minus the Killing form on $\mathfrak{so}(n)$ and $\mathbb{I}$ is the inertia matrix. Thus the metric is

$$\mu = \langle \mathbb{I}., . \rangle + \langle ., . \rangle_V.$$

(Units such that mass and radius equal to 1.)

Left vs. Right

Let $\widetilde{\mathcal{A}} : S \times \mathfrak{so}(n)_R \to V$, $(s, \widetilde{u}) \mapsto -\mathrm{pr}_V(\widetilde{u}.e_n)$. Then

$$\widetilde{\mathcal{D}} = \{(s, \widetilde{u}, x, -\widetilde{\mathcal{A}}(\widetilde{u}))\}.$$

In the *left* trivialization this becomes

$$\begin{aligned} \mathcal{A} : TS = S \times \mathfrak{so}(n) &\longrightarrow S \times \mathfrak{so}(n)_R \longrightarrow V \\ (s, u) &\longmapsto (s, \mathrm{Ad}(s)u) \longmapsto -\mathrm{pr}_V\big((\mathrm{Ad}(s)u).e_n\big), \end{aligned}$$

and $\widetilde{\mathcal{D}}$ translates to

$$\mathcal{D} = \{(s, u, x, -\mathcal{A}_s(u))\} \subset TS \times TV = TQ$$

which is a (*right* invariant) non-integrable distribution on Q.

The Chaplygin ball is the non-holonomic system described by the triple $(Q, \mathcal{D}, \frac{1}{2}||\cdot||_\mu^2)$. What about symmetries?

Symmetries

$(Q, \mathcal{D}, \frac{1}{2}||\cdot||_\mu^2)$ is a G-Chaplygin system with $G = V$.

(1) $\mathcal{A} : TS \to V$ is a connection form on the principal fiber bundle $V \hookrightarrow Q \twoheadrightarrow S$.

(2) $\mathcal{A}$ is right invariant.

The group $H = \{h \in S : h.e_n = e_n\} = \mathrm{SO}(n-1)$ acts through *two* different actions on Q:

(3) The l-action: $l_h(s, x) = (hs, x)$. This action generates *internal* symmetries: $\mathcal{A}\zeta_Y^l = \widetilde{\mathcal{A}}Y = 0$ for all $Y \in \mathfrak{h}$.

(4) The d-action: $d_h(s, x) = (hs, hx)$. This action generates *external* symmetries. $\mathcal{A}(hs, u) = h.\mathcal{A}(s, u)$ for all $h \in H$. Thus $\mathcal{D}$ is invariant under the d-action.

Notice that $\mathcal{A}$ is never the mechanical connection associated to μ, and $\mathrm{Curv}^{\mathcal{A}} = d\mathcal{A} \neq 0$.

The compressed system

Identify $T^*S = TS$ via the induced metric μ_0. According to general results on compression of internal symmetries:

The compressed Hamiltonian is

$$\mathcal{H}_\mathrm{c}(s,u) = \tfrac{1}{2}\langle \mathbb{I}u, u\rangle + \tfrac{1}{2}\langle \mathcal{A}_s(u), \mathcal{A}_s(u)\rangle_V$$

which is H-invariant. The compressed almost symplectic form is

$$\Omega_\mathrm{nh} = \Omega^S - \langle J_V \circ \mathrm{hl}^{\mathcal{A}}, \mathrm{Curv}_0^{\mathcal{A}}\rangle = \Omega^S + \langle \mathcal{A}, d\mathcal{A}\rangle_V$$

which is also H-invariant. The dynamics are given by X_nh:

$$i(X_\mathrm{nh})\Omega_\mathrm{nh} = d\mathcal{H}_\mathrm{c}.$$

Finally, there is a conserved quantity:

$$J_H : TS \to \mathfrak{h}^*$$

the standard momentum map. What about reduction?

Problem: $i(\zeta_Y)\Omega_\mathrm{nh} \neq d\langle J_H, Y\rangle$ whence $\Omega_\mathrm{nh}|J_H^{-1}(\lambda)$ is not horizontal for $J_H^{-1}(\lambda) \twoheadrightarrow J_H^{-1}(\lambda)/H_\lambda$.

Truncation

Let $\omega \in \Omega^1(S, \mathfrak{h})$ be the connection on $H \hookrightarrow S \twoheadrightarrow S^{n-1}$ associated to the biinvariant metric on S. Let $L = \omega : TS \to \mathfrak{h}$ viewed as a function, and $\mathrm{Curv}^\omega \in \Omega^2(S, \mathfrak{h})$ the curvature form.

Theorem (S.H. + L. Garcia-Naranjo, J. Geom. Mech. **1**(1))

Let

$$\widetilde{\Omega} := \Omega^S - \langle L, \mathrm{Curv}^\omega \rangle.$$

Then the system $(TS, \widetilde{\Omega}, \mathcal{H}_c)$ *satisfies:*

(1) $\widetilde{\Omega}$ *is non-degenerate and* H*-invariant.*

(2) $i(X_{nh})\widetilde{\Omega} = d\mathcal{H}_c$.

(3) $i(\zeta_Y)\widetilde{\Omega} = d\langle J_H, Y \rangle$ *for all* $Y \in \mathfrak{h}$.

For (2) one uses an $\langle ., . \rangle$-ONB Y_α, Z_a that is adapted to the decomposition $\mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ and *right* extends this to a frame on S. Then

$$\mathcal{A}_s(u) = -\sum \langle \mathrm{Ad}(s)^{-1} Z_a, u \rangle e_a =: -\sum \eta_s^a(u) e_a.$$

Corollary

$(TS, \widetilde{\Omega}, \mathcal{H}_c)$ can be reduced to an almost Hamiltonian system on

$$J_H^{-1}(\mathcal{O})/H \cong TS^{n-1} \times_{S^{n-1}} (S \times_H \mathcal{O})$$

where the isomorphism depends on μ_0, and $\mathcal{O} \subset \mathfrak{h}^*$. In particular, $\mathcal{O} \hookrightarrow J_H^{-1}(\mathcal{O})/H \to TS^{n-1}$.

Can also do point reduction: $J_H^{-1}(\mathcal{O})/H = J_H^{-1}(\lambda)/H_\lambda$ for $\lambda \in \mathcal{O}$.

Corollary

When $\mathbb{I} = 1$, Chaplygin's ball problem is Hamiltonian after reduction by H.

Proof.

In this case $L = J_H$. □

In fact, this is not surprising: $X^{\mathcal{M}} = X_{\mathcal{H}}$. However, the space $\mathcal{D}$ does not have a symplectic interpretation, and $\mathcal{D}/V$ is *not* a symplectic quotient.

Hamiltonization for $n = 3$

Let $n = 3$, then for $\lambda \in \mathfrak{h}^* = \mathbb{R}$

$$J_H^{-1}(\lambda)/S^1 \cong TS^2 \times \{\lambda\} = TS^2,$$

and the reduced almost symplectic form is

$$\text{canonical} + \text{magnetic} + \text{semi-basic}$$

with the semi-basic part linear in the fibers. According to Chaplygin ($\sim$1900) there is a preserved measure with density $f : S^2 \to \mathbb{R}_{>0}$. From Chaplygin's multiplier theorem we conclude:

Theorem

Let $\lambda \in \mathfrak{h}^*$. Then $d(f\widetilde{\Omega})|J_H^{-1}(\lambda) = 0$. (Conformally closed)

This theorem is due to Borisov and Mamaev (2001, 2005), albeit in a different setting: Using Euler angles and Lagrange multipliers they showed that after an f-dependent *time reparametrization*, $d\tau = fdt$, the equations of motion for the Chaplygin ball system can be written with respect to a Poisson bracket on $\mathbb{R}^3 \times \mathbb{R}^3$.

Part III: Chaplygin systems via semisimple Lie groups

Let G be a real semisimple Lie group with Lie algebra $\mathfrak{g}$ and Killing form B. Consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let $G \cong K \times \mathfrak{p}$, $g = k \exp x \leftarrow\!\shortmid (k, x)$ be the corresponding decomposition of the group. Thus:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, and put $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ and $M = Z_K(\mathfrak{a})$. Fix also a regular element $w \in \mathfrak{m}$. (Thus $Z_K(w) = M$ and $\mathrm{ad}(w) : \mathfrak{m}^{\perp} \cong \mathfrak{a}^{\perp}$.) The configuration space is:

$$Q := K \times V \text{ where } V := \mathfrak{a}^{\perp} \cong \mathfrak{p}/\mathfrak{a}.$$

The (natural kinetic energy) Lagrangian is

$$L = \tfrac{1}{2}\langle \mathbb{I}u, u\rangle + \tfrac{1}{2}\langle x', x'\rangle$$

where $\langle ., . \rangle = -B(., \theta .)$ is the usual pos. def. inner product. The Lagrangian is *left*-invariant since we identify $TK = K \times \mathfrak{k}$ via the *left* multiplication, $u = s^{-1}s'$.

The constraint distribution and its symmetries

The distribution is

$$\mathcal{D} = \{(s, u, x, -\mathcal{A}_s(u))\} \subset TK \times TV$$

where

$$\mathcal{A} : TK \ni (s, u) \longmapsto -[\mathrm{Ad}(s)u, w] \in V.$$

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(2) $\mathcal{A}$ is right invariant.

The group $M = \{h \in K : \mathrm{Ad}(h)w = w\}$ acts through *two* different actions on Q:

(3) The l-action: $l_h(s, x) = (hs, x)$. This action generates *internal* symmetries: $\mathcal{A}\zeta^l_Y = 0$ for all $Y \in \mathfrak{m}$. $(\zeta^l_Y(s) = \mathrm{Ad}(s^{-1}).Y)$

(4) The d-action: $d_h(s, x) = (hs, hx)$. This action generates *external* symmetries. $\mathcal{A}(hs, u) = h.\mathcal{A}(s, u)$ for all $h \in M$. Thus $\mathcal{D}$ is invariant under the d-action.

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which is M-invariant. The compressed almost symplectic form is

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which is also M-invariant. The dynamics are given by X_nh:

$$i(X_\mathrm{nh})\Omega_\mathrm{nh} = d\mathcal{H}_\mathrm{c}.$$

According to the NH-Noether theorem the standard MoMap

$$J_M : TK \to \mathfrak{m}^*$$

is a conserved quantity. Reduction? Preserved measure? Hamiltonization?

Example: $G = \mathrm{SO}(p,q)_o$

Let $G = \mathrm{SO}(p,q)_o$ with $p \geq q$. Then:

$$K = \{\mathrm{diag}(A, D) : A \in \mathrm{SO}(p), D \in \mathrm{SO}(q)\}$$

$$\mathfrak{p} = \{\begin{pmatrix} 0_{p\times p} & b \\ b^t & 0_{q\times q} \end{pmatrix} : b \in \mathfrak{gl}(p \times q, \mathbb{R})\}$$

$$\begin{aligned} \mathfrak{a} &= \{\begin{pmatrix} 0_{p\times p} & b \\ b^t & 0_{q\times q} \end{pmatrix} : b \text{ has only lower antidiagonal non-zero}\} \\ &= \mathbb{R}^q \\ M &= \{\mathrm{diag}(\mathrm{SO}(p-q), \theta_q, \ldots, \theta_1, \theta_1, \ldots, \theta_q) : \theta_i = \pm 1, \Pi\theta_i = 1\} \\ &= \mathrm{SO}(p-q) \times \{\pm 1\}^{q-1} \end{aligned}$$

Therefore,

$$\begin{aligned} K/M &= (\mathrm{SO}(p)/\mathrm{SO}(p-q) \times \mathrm{SO}(q))/\{\pm 1\}^{q-1} \\ &\cong V(q,p) \times \mathrm{SO}(q)/\{\pm 1\}^{q-1}. \end{aligned}$$

Special case $q = 1$, $p = n \geq 3$

In this case:

$$
\begin{aligned}
K &= \mathrm{SO}(n) \times \{1\} \\
\mathfrak{p} &= \{ \begin{pmatrix} 0_{p\times p} & b \\ b^t & 0 \end{pmatrix} : b \in \mathfrak{gl}(n \times 1, \mathbb{R}) = \mathbb{R}^n \} \\
\mathfrak{a} &\cong \mathbb{R}^1 \text{ and } V = \mathfrak{a}^\perp \cong \mathbb{R}^{n-1} \\
M &\cong \mathrm{SO}(n-1) \\
\text{Thus } Q &\cong \mathrm{SO}(n) \times \mathbb{R}^{n-1}
\end{aligned}
$$

and $\mathcal{A}_s(u) = -[\mathrm{Ad}(s)u, w]$ can be identified with the previous $\mathcal{A} = -\mathrm{pr}_V(\mathrm{Ad}(s)u.e_n) \in \mathbb{R}^{n-1}$.

Moreover,

$$K/M = V(1, n) = S^{n-1}$$

whence we recover the n-dimensional Chaplygin ball. (The Lagrangian L also identifies in the expected way.)

Describing the system

To write $\mathcal{A}$ in terms of a suitable *right* invariant coframe, let $\Sigma \subset \mathfrak{a}^*$ denote the set of restricted roots associated to $(\mathfrak{g}, \mathfrak{a})$ and let $\Sigma^+ \subset \Sigma$ be a set of positive roots. Then we write the root space decomposition as

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda.$$

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$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda.$$

Choose an ONB of $\mathfrak{k}$ (*not* consisting of root vectors)

$$Y_\alpha,\ \alpha = 1, \ldots, \dim \mathfrak{m}, \quad Z_{(\lambda,a)},\ \lambda \in \Sigma^+,\ a = 1, \ldots, \dim \mathfrak{g}_\lambda$$

that is adapted to the decomposition $\mathfrak{m} \oplus \mathfrak{m}^\perp$, and an ONB of V

$$e_{(\lambda,a)},\ \lambda \in \Sigma^+,\ a = 1, \ldots, \dim \mathfrak{g}_\lambda$$

such that

$$\mathrm{ad}(w) Z_{(\lambda,a)} = \lambda(w) e_{(\lambda,a)} \text{ and } \mathrm{ad}(w) e_{(\lambda,a)} = \lambda(w) Z_{(\lambda,a)} \text{ for } w \in \mathfrak{a}.$$

Extend the dual basis of Y_α, $Z_{(\lambda,a)}$ to a *right* invariant coframe $\rho^\alpha, \eta^{(\lambda,a)}$ of K.

It follows that:

$$\mathcal{A} = -\sum_{\lambda\in\Sigma_+,\, a=1,\dots \dim \mathfrak{g}_\lambda} \lambda(w)\eta^{(\lambda,a)} e_{(\lambda,a)}$$

whence

$$\mathcal{H}_c(s,u) = \tfrac{1}{2}\langle \mathbb{I}u, u\rangle + \tfrac{1}{2}\sum \lambda(w)^2 \langle s^{-1} Z_{(\lambda,a)}, u\rangle^2$$

and

$$\Omega_{nh} = \Omega^{TK} + \sum \lambda(w)^2 \langle s^{-1} Z_{(\lambda,a)}, u\rangle d\eta^{(\lambda,a)}.$$

Moreover, there is a preserved measure: consider $\mu_0(s) : \mathfrak{k} \to \mathfrak{k}$,

$$\mu_0(s) = \mathbb{I} + \sum \lambda(w)^2 \eta_s^{(\lambda,a)} \otimes s^{-1} Z_{(\lambda,a)}$$

Theorem

Let $d = \dim K$ *and* $f = (\det \mu_0)^{-\frac{1}{2}}$. *Then*

$$L_{X_{nh}}(f(\Omega^{TK})^d) = 0.$$

For the nD Chaplygin ball ($G = SO(n,1)$) this is due to Fedorov and Kozlov (1995).

Reduction of internal symmetries: Truncation

M acts on $(TK, \Omega_{\mathsf{nh}}, \mathcal{H}_{\mathsf{c}})$ and $J_M : TK \to \mathfrak{m}^*$ is a conserved quantity. Reduction? Problem:

$$i(\zeta_Y)\Omega_{\mathsf{nh}} \neq d\langle J_M, Y\rangle \text{ for } Y \in \mathfrak{m}$$

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Reduction of internal symmetries: Truncation

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Theorem

Let

$$\widetilde{\Omega} = \Omega^{TK} - \tfrac{1}{2}\sum \lambda(w)^2 c^{\alpha}_{(\lambda,a)(\lambda,b)}\langle s^{-1}Y_\alpha, u\rangle \eta^{(\lambda,a)} \wedge \eta^{(\lambda,b)} + \tfrac{1}{2}\sum \lambda(w)^2 c^{(\lambda,a)}_{(\mu,b)(\nu,c)}\langle s^{-1}Z_{(\lambda,a)}, u\rangle \eta^{(\mu,b)} \wedge \eta^{(\nu,c)}.$$

Then

(1) $\widetilde{\Omega}$ *is non-deg. and* M*-invariant, and* $i(X_{\mathrm{nh}})\widetilde{\Omega} = d\mathcal{H}_{\mathrm{c}}$.

(2) $i(\zeta_Y)\widetilde{\Omega} = d\langle J_M, Y\rangle$ *for all* $Y \in \mathfrak{m}$.

The main step in the proof is to reckognize that $i(X_{\mathrm{nh}})\langle \mathcal{A}, d\mathcal{A}\rangle$ is horizontal.

Hamiltonization at 0 momentum

Thus we may replace $(TK, \Omega_{\mathrm{nh}}, \mathcal{H}_{\mathrm{c}})$ by $(TK, \widetilde{\Omega}, \mathcal{H}_{\mathrm{c}})$ and reduce the latter with respect to M and the MoMap J_M. Hamiltonization? When $\dim K/M = 2$ the reduced system is Hamilonizable by Chaplygin's multiplier theorem. For $\dim K/M > 2$ we can use the higher dimensional multiplier theorem to obtain a criterium for Hamiltonizability at $J_M = 0$. When

$$|\Sigma_+| = 1$$

(e.g., the Chaplygin ball case $G = \mathrm{SO}(n,1)$) then this becomes:

Theorem

The reduction of $(TK, \widetilde{\Omega}, \mathcal{H}_{\mathrm{c}})$ is Hamiltonizable (i.e., $d(f^{\frac{1}{m-1}}\widetilde{\Omega}_{\mathrm{reduced}}) = 0$, $m = \dim K/M$) at $J_M = 0$ iff

$$(m-1)\langle [Z_a, Z_b], s\mu_0^{-1}s^{-1}Z_c\rangle = \sum_d \langle [Z_a, Z_d], s\mu_0^{-1}s^{-1}Z_d\rangle \delta_{bc} - \sum_d \langle [Z_b, Z_d], s\mu_0^{-1}s^{-1}Z_d\rangle \delta_{ac}$$

The n-D Chaplygin ball

For the Chaplygin ball with $\mathfrak{k} = \mathfrak{so}(n) = \mathbb{R}^n \wedge \mathbb{R}^n$, assuming that $\mathbb{I}$ preserves $\mathfrak{m} \oplus \mathfrak{m}^\perp$, this implies that

$$\mathbb{I} e_i \wedge e_j = \frac{a_i a_j}{\lambda(w) - a_i a_j} e_i \wedge e_j \text{ where } a_i \in \mathbb{R} \text{ and } 0 < a_i a_j < \lambda(w) \ \forall i, j$$

which is the inertia tensor of Jovanovic (2009) who also showed that the reduced system is integrable for this choice of inertia tensor.

Non-Hamiltonizability

When $G = \mathrm{SL}(n, 3)$ then the corresponding Chaplygin system (associated to $w = \mathrm{diag}(w_1 > w_2 > w_3)$) is *not* Hamiltonizable for *any* choice of inertia tensor $\mathbb{I}$ – including the homogeneous case.

When, e.g., $w_1 = w_2$ then one recovers the 3-D Chaplygin ball.