Non-holonomic diffusions

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1. Introduction

A non-holonomic system $(Q, \mathcal{D}, \mathcal{L})$ consists of a configuration space Q, a Lagrangian $\mathcal{L}$ and a constraint distribution $\mathcal{D} \subset TQ$ specifying the totality of all allowed velocities. The equations of motion are determined by the corresponding Lagrange-d'Alembert principle. An example is the snakeboard on a horizontal plane: $Q = S^1 \times S^1 \times S^1 \times \mathrm{SE}(2)$, $\mathcal{L}$ is the kinetic energy Lagrangian, and the constraints $\mathcal{D}$ are specified by disallowing sliding of the wheels. See [5]. Imagine the snakeboard is shrunk to microscopic size and bombarded by molecular bullets uniformly from all sides in the plane where it moves. Assuming that the constraints are still in force this setup gives rise to a *stochastic non-holonomic system* and a process which describes the corresponding (stochastic) motion is referred to as a *non-holonomic diffusion*.

2. Basic idea of the construction

The geometric setting of stochastic Hamiltonian systems is due to [4]. They have defined a *stochastic Hamiltonian system* to be a set (M, H, Y) where M is a Poisson manifold, $H = (H^i) : M \to \mathbb{R}^k$ is an $\mathbb{R}^k$-valued Hamiltonian and $Y = (Y_i) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^k$ is a semi-martingale. The equations of motion are determined by the Stratonovich stochastic differential equation

$$\delta\Gamma = \sum X_{H^i}(\Gamma)\, \delta Y_i \tag{1}$$

where X_{H^i} are the Hamiltonian vectorfields. The solution process $\Gamma : \Omega \times \mathbb{R}_+ \to M$, which exists and is unique up to explosion time, is again a semi-martingale.

To apply this set-up to non-holonomic mechanics the basic idea is that, for a wide class of non-holonomic systems, the equations of motion may be encoded in terms of a standard Hamiltonian vector field together with a projection operator which stems from the constraint forces. This is the Hamiltonian description of a non-holonomic mechanical system $(Q, \mathcal{D}, \mathcal{L} = T - V)$ where T is the kinetic energy of a metric μ. The constraint force projection is the projection operator

$$P : TTQ|\mathcal{D} \longrightarrow \mathcal{C} := \{\xi \in TTQ|\mathcal{D} : T\tau.\xi \in \mathcal{D}\}$$

along the annihilator of $\mathcal{C}$ for the canonical symplectic form on $TQ \cong_\mu T^*Q$.

Applying P to (1) yields a new Stratonovich equation

$$\delta\Gamma^{\mathrm{nh}} = \sum P(\Gamma^{\mathrm{nh}}) X_{H^i}(\Gamma^{\mathrm{nh}})\, \delta Y_i$$

the solution semi-martingale Γ^{nh} of which is thought of as describing a *stochastic non-holonomic system*. See [1, 2].

3. Constrained Brownian motion

Let (Q, μ) be a Riemannian manifold acted upon by an isometry group G in a free and proper fashion, $\rho : \mathcal{F} \to Q$ be the orthonormal frame bundle, and $\mathrm{Hor}^\mu \subset T\mathcal{F}$ the horizontal space corresponding to the Levi-Civita connection ∇^μ. Let $W = (W^i)$ be Brownian motion in $\mathbb{R}^k$.
To construct BM in (Q, μ):

- There are globally defined vfs -"the canonical horizontal vfs"- $L_1, \dots, L_k \in \mathfrak{X}(\mathcal{F}, \mathrm{Hor}^\mu)$.
- Define $H^i : T^*\mathcal{F} \to \mathbb{R}$, $(u, \eta) \mapsto \langle \eta_u, L_i(u) \rangle$.
- If $\Gamma : \Omega \times \mathbb{R}_+ \to T^*\mathcal{F}$ solves $\delta\Gamma = X_{H^i}(\Gamma)\delta W^i$ then $\rho \circ \tau \circ \Gamma$ is BM in (Q, μ).

Want to apply a constraint force projection to X_{H^i}. Lift $(Q, \mathcal{D}, \mu)$ and G-action to $(\mathcal{F}, \mathcal{D}^{\mathcal{F}}, \mu^{\mathcal{F}})$:

- μ can be lifted to the Sasaki-Mok metric $\mu^{\mathcal{F}}$ on $\mathcal{F}$, and then $T\mathcal{F} = T^*\mathcal{F}$ via $\mu^{\mathcal{F}}$.
- G acts on $\mathcal{F}$ and preserves $\mu^{\mathcal{F}}$.
- $\mathcal{D}^{\mathcal{F}} \cong (\mathcal{F} \times_Q \mathcal{D}) \oplus \mathrm{Ver}(\rho) \subset (\mathcal{F} \times_Q TQ) \oplus \mathrm{Ver}(\rho) \cong T\mathcal{F}$.

Thus we obtain $TT\mathcal{F}|\mathcal{D}^{\mathcal{F}} = \mathcal{C}^{\mathcal{F}} \oplus (\mathcal{C}^{\mathcal{F}})^{\Omega_{\mathcal{F}}}$ and the constraint force projector

$$P^{\mathcal{F}} : TT\mathcal{F}|\mathcal{D}^{\mathcal{F}} \longrightarrow \mathcal{C}^{\mathcal{F}}.$$

If $\Gamma : \Omega \times \mathbb{R}_+ \to \mathcal{D}^{\mathcal{F}}$ solves

$$\delta\Gamma = P^{\mathcal{F}}(\Gamma) X_{H^i}(\Gamma)\delta W^i$$

then $\rho \circ \tau \circ \Gamma =: \Gamma^{\mathrm{nh}}$ is CONSTRAINED BM in Q.

Theorem 3.1 ([2]). *Let $\Pi : TQ = \mathcal{D} \oplus \mathcal{D}^\perp \to \mathcal{D}$ be the orthogonal projection and $u = (u^i)$ a local ONF. Then Γ^{nh} is a diffusion in Q with generator*

$$\tfrac{1}{2}(\Pi u^i)(\Pi u^i) - \tfrac{1}{2}\Pi\nabla^\mu_{\Pi u^i} u^i$$

which is a 2^{nd}-order DO acting on $C^\infty_{\mathrm{cp}}(Q)$.

Corollary 3.2. *It follows that Γ^{nh} is a martingale for the non-holonomic connection ∇^{nh}.*

Corollary 3.3. *If $(Q, \mathcal{D}, \mathcal{L})$ is invariant under a free and proper Lie group action then Γ^{nh} descends to 'drifted BM' on the quotient Q/G.*

4. Conclusions for Chaplygin type constraints

Suppose the constraints $\mathcal{D}$ are of G-Chaplygin type such that $\mathcal{D}$ is a principal bundle connection on $Q \twoheadrightarrow Q/G =: M$ for a free and proper action of an isometry Lie goup G on (Q, μ). Then Γ^{nh} induces drifted BM Γ^M with drift $b \in \mathfrak{X}(M)$ in M.

Theorem 4.1 ([2]). *Assume M is compact (for simplicity). Then the following are equivalent.*

- *The deterministic system $X^{\mathrm{nh}}_{\mathcal{H}_0}$ preserves a volume $\mathcal{N}\Omega^m_{TM}$ on TM.*
- *The non-holonomic diffusion Γ^M is time reversible in the sense of [3].*
- *$\beta = \check{\mu}_0(b)$ is exact, and then $\beta = d(\log \mathcal{N})$.*
- *The entropy production rate of Γ^M is 0. – purely probabilistic condition.*

The *candidate* $\mathcal{N}$, the equilibrium density which exists and is unique for compact M, is characterized by

$$A^*\mathcal{N} = \tfrac{1}{2}\Delta^\mu\mathcal{N} - \mathrm{div}_{\mathrm{vol}_\mu}(\mathcal{N}b) = 0$$

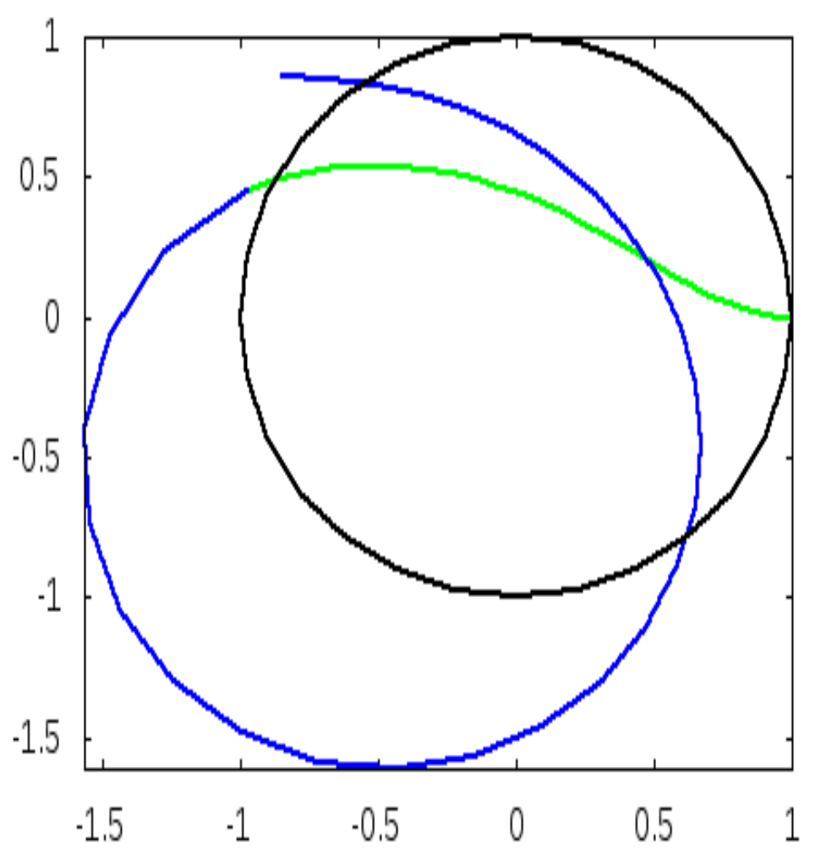
up to a multiplicative constant. Compactness is satisfied for important examples such as the two-wheeled robot, the Chaplygin ball, or the snakeboard.

5. The two-wheeled mobile robot

The two-wheeled planar robot is an example which fits very well into this scheme. It is a G-Chaplygin system with $G = \mathrm{SE}(2)$. One can show that it does not admit a preserved measure whence the associated diffusion is not time-reversible.

We have also considered the motion planning problem for this system. The set-up is taken from [6]. Here the wheels of the robot are subject to a Gaussian white noise, and the goal is to make the most probable path of the cart follow a predefined nominal curve. We can find this path by using the so-called Onsager-Machlup function, together with the formula for the drift vectorfield. See [2].

If the nominal curve is a circle and the control input is such that the *deterministic* cart follows this curve then the most probable motion of the *stochastically perturbed* system has the following form:



The nominal curve C is black. The part of the curve where the wheel speeds are increasing is in green while the part where they are decreasing is in blue.

The idea for motion planning is now to use the explicit formula for the most probable path of the perturbed vehicle to design the control input (i.e., the wheel speeds) such that the most probable path of the noisy vehicle follows the nominal trajectory (up to a pre-defined error).

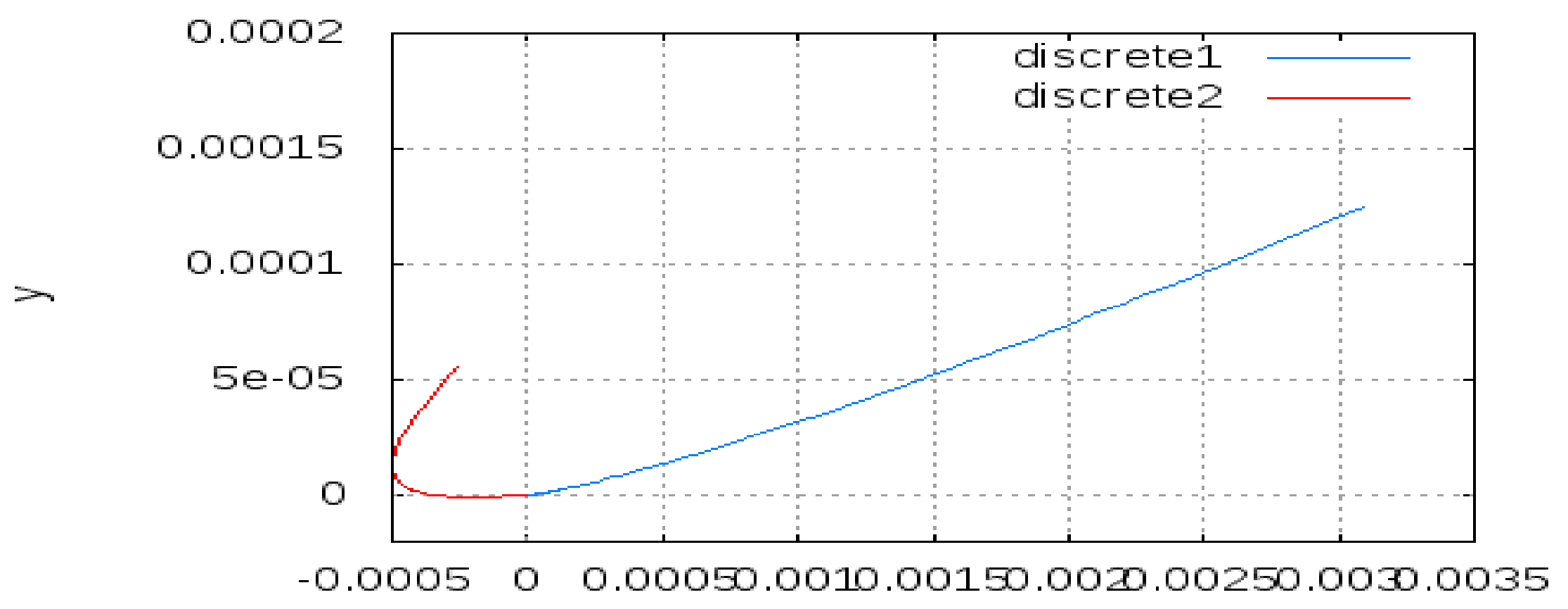
6. The microscopic snakeboard

Another example where we have found a similar phenomenon is the snakeboard. This is a variant of the skateboard where one may generate forward motion without touching the ground. See [5]. This effect is due to the non-holonomic momentum equation. The control and motion planning for this system are well-understood and known to allow for *gaits*.

The ObliqO version of the snakeboard.

In [2] we have considered the snakeboard on a horizontal plane subject to a translational and rotational Brownian motion jiggling of the plane. Equivalently one may picture a microscopic snakeboard under random bombardment of molecular particles.

Again we obtain the drift from general principles and use Onsager-Machlup theory to find the most probable path to be followed by the stochastic snakeboard. For the present example, and omitting higher order terms, this amounts to a system of first order ODE's. Using a Runge-Kutta scheme one obtains the following picture:



This is the plot up to time $T = 1$. The blue curve is the center of mass motion of the gait-controlled deterministic snakeboard. The red curve is the most probable center of mass motion of the stochastically perturbed system with the same deterministic input.

These curves differ quite strongly, and we emphasize that this effect is caused by the very special manner in which the noise couples with the non-holonomic constraints, a phenomenon which is not present, in this form, for the perturbation of Hamiltonian systems.

References

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