To construct BM in Theorem 3.1 ρ Γ : Ω TT (a) characterized by which is a the solution semi-martingale. Applying a candidate local ONF. Then along the annihilator of for the canonical symplectic form on TTQ. Letting i = 1 yields a new Stratonovich equation the solution semi-martingale of which is thought of as describing a stochastic non-holonomic system. See [1, 2].

3. Constrained Brownian motion

Let (Q, µ) be a Riemannian manifold actuated upon by an isometry group G in a free and proper fashion. ρ, F → Q be the orthonormal frame bundle, and HDP, TF the horizontal space corresponding to the Levi-Civita connection Γ. Let i = 1 be Brownian motion in Rn.

To construct BM in (Q, µ) there are globally defined vfs “the canonical horizontal vfs” L1, . . . , Lk ∈ X(F, HorF).

Define H: TF → R, w(u, a) → (u, Lk(a)). If Γ × TF solves δ t − Xπ(Γ)(δ t µ) then is BM in (Q, µ).

Want to apply a constraint force projection to Xπ(Γ). Lift (Q, DQ, G) to (F, DF, µF); ρ can be lifted to the Sasaki-Mok metric µF on F, and then F → TF via µF. G acts on F and preserves µF.

δ t ∼ (F × Q) ∆Q → (F × Q) ∆Q by Ver(µ) = Ver(µF) Ver(µ) ∼ TF.

Thus we obtain TFΓDF = DQγ(DF) and the constraint force projector µF(Γ F ∇F) − γ(DF).

If i = 1 × TF solves µF(Γ F ∇F) − γ(DF).

Theorem 3.1 ([2]). Let TQ ⊆ DQ ⊆ DQ → DQ be the orthogonal projection and s = (a) a local ONF. Then is a diffusion (Q, µ) with generator

which is a L2 Sobolev acting on (C2(Q) 2).

Corollary 3.2. If it follows that i is a martingale for the non-holonomic connection Γ.

Corollary 3.3. If (Q, D, L) is invariant under a free and proper Lie group action then deforms to “drifted BM” on the quotient Q/G.

4. Conclusions for Chaplygin type constraints

Suppose the constraints D are of Chaplygin type such that D is a principal bundle connection on (Q) G/Q for a free and proper action of an isometry group G on (Q, µ). Then induces drifted BM on with drift

Theorem 4.1 ([2]). Assume X is compact (for simplicity). Then the following are equivalent.

The deterministic system X preserving a volume X of TM.

The non-holonomic diffusion Γ is time reversible in the sense of [3].

The entropy production rate of Γ is purely probabilistic condition.

The candidate X, the equilibrium density which exists and is unique for compact X, is characterized by dp(X, N) = 0 up to a multiplicative constant. Maxima is satisfied for important examples such as the two-wheeled robot, the Chaplygin ball, or the snakeboard.

5. The two-wheeled mobile robot

The two-wheeled planar robot is an example which fits very well into this scheme. It is a Chaplygin system with G = SE(2). One can show that it does not admit a preserved measure whence the associated diffusion is not time-reversible.

We have also considered the motion planning problem for this system. The set-up is taken from [6]. Here the wheels of the robot are subject to a Gaussian white noise, and the goal is to find the most probable path of the cart follow a predefined nominal curve. We can find this path by using the so-called Onsager-Machlup function, together with the formula for the drift vectorfield. See [2].

If the nominal curve is a circle and the control input is such that the deterministic cart follows this curve then the most probable motion of the stochastically perturbed system has the following form:

6. The microscopic snakeboard

Another example where we have found a similar phenomenon is the snakeboard. This is a model of the skateboard where one may generate forward motion without touching the ground. See [5]. This effect is due to the non-holonomic momentum equation. The control and motion planning for this system are well-understood and known to allow for gaits.

The OblOQ version of the snakeboard. In [2] we have considered the snakeboard on a horizontal plane subject to a translational and rotational Brownian motion juggling of the plane. Equivalently one might picture a microscopic snakeboard under random bombardment of molecular particles. Again we obtain the drift from general principles and use Onsager-Machlup theory to find the most probable path to be followed by the stochastic snakeboard. For the present example, and omitting higher order terms, this amounts to a system of first order ODE’s. Using a Runge-Kutta scheme one obtains the following picture:

This is the plot up to time T = 1. The blue curve is the center of mass motion of the gait-controlled deterministic snakeboard. The red curve is the most probable center of mass of the stochastically perturbed system with the same deterministic input. These curves differ quite strongly, and we emphasize that this effect is caused by the very special manner in which the noise couples with the non-holonomic constraints, a phenomenon which is not present, in this form, for the perturbation of Hamiltonian systems.

References