The geometry of non-holonomic diffusions

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Contents

Stochastic differential equations

We are concerned with a continuous stochastic process Γ with values in a mfd M. This means that there is a probability space $(\Omega, \mathcal{F}, P)$ and $\Gamma : \Omega \times \mathbb{R}_+ \to M$.

To make sense of a dynamical equation for Γ we need the following:

A Stratonovich operator from $T\mathbb{R}^k$ to TM is a family of linear maps

$$\mathcal{S}_{(x,y)} : T_x\mathbb{R}^k \longrightarrow T_y M$$

which depend smoothly on $(x, y) \in \mathbb{R}^k \times M$; so $\mathcal{S}$ is a section of $T^*\mathbb{R}^k \otimes TM$.

Consider vfs $X_0, X_1, \ldots, X_k$ on M and a stochastic process

$$Y : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}_+ \times \mathbb{R}^k, \quad (t, \omega) \longmapsto (t, W_t(\omega))$$

where $W = (W^i) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^k$ is a BM in $\mathbb{R}^k$.

The associated STRATONOVICH STOCHASTIC DIFFERENTIAL EQUATION is

$$\delta\Gamma = \mathcal{S}(Y,\Gamma)\delta Y := X_0(\Gamma)\delta t + X_i(\Gamma)\delta W^i.$$

A continuous and adapted process $\Gamma : \Omega \times \mathbb{R}_+ \to M$ is called a solution to this equ. if, for all $f \in C^\infty(M)$, in the Stratonovich sense:

$$f(\Gamma_t) - f(\Gamma_0) = \int_0^t (X_0 f)(\Gamma_s) ds + \int_0^t (X_i f)(\Gamma_s)\delta W_s^i.$$

Existence and uniqueness

Given an initial condition $\Gamma_0 = x \in M$, solutions exist and are unique. Moreover, the solution Γ is a diffusion (strong Markov property) in M – driven by k-dimensional BM.

Diffusions, martingales and BM

Suppose Γ solves $\delta\Gamma = \mathcal{S}(Y,\Gamma)\delta Y = X_0(\Gamma)\delta t + X_i(\Gamma)\delta W^i$.

Generator

The generator of Γ is the second order diff. op. A acting on $f \in C^\infty(M)$ by

$$Af = X_0 f + \frac{1}{2} X_i X_i f.$$

- If ∇ is a linear connection on $TM \to M$ then Γ is called a MARTINGALE in (M, ∇) if A is purely second order;
- ...this means that the ∇-DRIFT $X_0 + \frac{1}{2}\nabla_{X_i} X_i$ of the diffusion is 0.
- If (M, μ) is a Riemannian manifold then Γ is called BM in (M, μ) if $A = \frac{1}{2}\Delta_\mu$.

([Ikeda, Watanabe 1989] and [Emery, 1989])

Time reversible diffusions

Assume (for simplicity) that M is a compact Riemannian manifold and Γ a diffusion in M with

$$A = \frac{1}{2}\Delta^{\mu} + \frac{1}{2}b.$$

There is a function (fundamental solution) $p : \mathbb{R}_+ \times M \times M \to \mathbb{R}_{>0}$ such that the transition probability of Γ is given by

$$P(t,x,S) = \int_S p(t,x,y)\mathrm{vol}_\mu(y) \quad \text{for } S \subset M.$$

Let $\mathcal{N}$ denote the unique equilibrium density function, that is, $P(t,x,S) \longrightarrow \int_S \mathcal{N}\mathrm{vol}_\mu$ as $t \longrightarrow \infty$.

Γ is TIME REVERSIBLE if

$$p(t,x,y)\mathcal{N}(x) = p(t,y,x)\mathcal{N}(y).$$

(Detailed balance equation.)

Characterization of time reversible diffusions

The following are equivalent:

- Γ is time reversible;
- (Kolmogorov [1936]:) b is a gradient, and in this case

$$b = \mathrm{grad}^{\mu}(\log \mathcal{N})$$

  where $\mathcal{N}$ is the equilibrium density;
- (Qian, Qian, Tang [2002]:) The entropy production rate of Γ vanishes, i.e., $\lim_{t\to\infty} \frac{1}{t} H(P_{[0,t]}, P^{-}_{[0,t]}) = 0$.

The *candidate* $\mathcal{N}$, which exists and is unique for compact M, is characterized by

$$\int_M \mathcal{N} \mathrm{vol}_{\mu} = 1 \quad \text{and} \quad A^*\mathcal{N} = \frac{1}{2}\Delta^{\mu}\mathcal{N} - \mathrm{div}_{\mathrm{vol}_{\mu}}(\mathcal{N} b) = 0.$$

Part II: NH-systems: the (almost) Hamiltonian description

Let (Q, μ) be a Riemannian manifold and G a Lie group acting freely and properly on Q by isometries. Let $\pi : Q \twoheadrightarrow Q/G =: M$. We are interested in nh-systems of the type $(Q, \mathcal{D}, \mathcal{L})$ where $\mathcal{D} \subset TQ$ is the constraint subbundle, $\mathcal{L} = \frac{1}{2}|| \cdot ||_\mu^2$ is the Lagrangian, and $\mathcal{D}$ is G-invariant. Further:

- Assume: $TQ = \mathcal{D} + \mathrm{Ver}, \quad \mathrm{Ver} := \ker(T\pi : TQ \twoheadrightarrow TM)$;
- $\mathcal{S} := \mathcal{D} \cap \mathrm{Ver}, \quad \mathrm{Hor} := \mathcal{S}^\perp \cap \mathcal{D}, \quad \mathcal{U} := \mathcal{S}^\perp \cap \mathrm{Ver}$ are G-invariant;
- $TQ = \mathrm{Hor} \oplus \mathcal{S} \oplus \mathcal{U} = \mathcal{D} \oplus \mathcal{U}$;
- $TQ =_\mu T^*Q$ and $\mathcal{H} = \mathcal{L}$;
- $\mathcal{C} := \{\xi \in TTQ|\mathcal{D} : T\tau.\xi \in \mathcal{D}\}$
- (Bates, Sniatycki [93]:) $TTQ|\mathcal{D} = \mathcal{C} \oplus \mathcal{C}^\Omega$;
- Let $P : TTQ|\mathcal{D} \longrightarrow \mathcal{C}$ denote the projection along $\mathcal{C}^\Omega$.

Dynamics

The dynamics of $(Q, \mathcal{D}, \mathcal{L})$ are given by the vf

$$X_{\mathcal{H}}^{\mathcal{C}} := PX_{\mathcal{H}} \in \mathfrak{X}(\mathcal{D})$$

where $X_{\mathcal{H}}$ is the Hamiltonian vf.

Chaplygin reduction

Assume $\mathcal{S} = 0$. Then $\mathcal{D}/G = \mathrm{Hor}/G = TM$, $\mathcal{H}$ factors to $\mathcal{H}_0 : TM \to \mathbb{R}$, and $X_{\mathcal{H}}^{\mathcal{C}}$ projects to a vf. $X_{\mathcal{H}_0}^{\mathrm{nh}}$, which can be encoded in an almost Hamiltonian structure $(TM, \Omega_{\mathrm{nh}}, \mathcal{H}_0)$.

Preservation of volume

(Cantrijn, Cortes, de Leon, Martin de Diego [2002]:) $X_{\mathcal{H}_0}^{\mathrm{nh}}$ is volume preserving iff

$$\beta := \langle J(u_i), \mathrm{Curv}^{\mathcal{D}}(X_{\mathcal{H}_0}^{\mathrm{nh}}, u_i\rangle \text{ is exact;}$$

(u_i) ONF on M. If $\beta = d(\log \mathcal{N})$ then $\mathcal{N}\Omega_{TM}^m$ is preserved.

Part III: Non-holonomic diffusions

Stochastic Hamiltonian systems

(Lazaro-Cami, Ortega [2007]:) Consider a symplectic mfd. (N, ω), an $\mathbb{R}^{k+1}$-valued Hamiltonian $H = (H^i) : N \to \mathbb{R}^{k+1}$, and $Y = (t, W^i) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{k+1}$ as before. Then the STOCHASTIC HAMILTONIAN SYSTEM (N, ω, H, Y) is given by the Stratonovich equation

$$\delta\Gamma = \mathcal{S}_H(Y, \Gamma) := X_{H^i}(\Gamma)\delta Y^i.$$

Hamiltonian construction of BM

Let $(N, \omega) = (T^*Q, \Omega^Q)$, (Q, μ) Riemannian. Assume $TQ = Q \times \mathbb{R}^k$ and $u_1, \ldots, u_k$ a global ONF. Define

$$H_0 = -\tfrac{1}{2}\langle p, \nabla^\mu_{u_i} u_i\rangle \text{ and } H_i := \langle \_, u_i\rangle : T^*Q \to \mathbb{R}.$$

If Γ solves $\delta\Gamma = \mathcal{S}_H(Y, \Gamma)\delta Y$ then $\tau \circ \Gamma$ is BM in (Q, μ).

Constrained BM: $\delta\Gamma^{\mathrm{nh}} = P.\mathcal{S}_H(Y, \Gamma^{\mathrm{nh}}) := P.X_{H^i}(\Gamma^{\mathrm{nh}})\delta Y^i.$

Constrained BM for general (Q, μ)

Let $\rho : \mathcal{F} \to Q$ be the orthonormal frame bundle and $\mathrm{Hor}^\mu \subset T\mathcal{F}$ the horizontal space corresponding to the Levi-Civita connection ∇^μ. Let $W = (W^i)$ BM in $\mathbb{R}^k$.

- There are globally defined vfs - "the canonical horizontal vfs" - $L_1, \dots, L_k \in \mathfrak{X}(\mathcal{F}, \mathrm{Hor}^\mu)$ which play the role of u_i.
- Define $H^i : T^*\mathcal{F} \to \mathbb{R}$, $(u, \eta) \mapsto \langle \eta_u, L_i(u) \rangle$.
- If $\Gamma : \Omega \times \mathbb{R}_+ \to T^*\mathcal{F}$ solves $\delta\Gamma = X_{H^i}(\Gamma)\delta W^i$ then $\rho \circ \tau \circ \Gamma$ is BM in (Q, μ).

Want to apply a constraint force projection to X_{H^i}. Lift $(Q, \mathcal{D}, \mu)$ and G-action to $(\mathcal{F}, \mathcal{D}^\mathcal{F}, \mu^\mathcal{F})$.

- μ can be lifted to the Sasaki-Mok metric $\mu^\mathcal{F}$ on $\mathcal{F}$, and then $T\mathcal{F} = T^*\mathcal{F}$ via $\mu^\mathcal{F}$.
- G acts on $\mathcal{F}$ and preserves $\mu^\mathcal{F}$.
- $\mathcal{D}^\mathcal{F} \cong (\mathcal{F} \times_Q \mathcal{D}) \oplus \mathrm{Ver}(\rho) \subset (\mathcal{F} \times_Q TQ) \oplus \mathrm{Ver}(\rho) \cong T\mathcal{F}$.

Thus we obtain $TT\mathcal{F}|\mathcal{D}^{\mathcal{F}} = \mathcal{C}^{\mathcal{F}} \oplus (\mathcal{C}^{\mathcal{F}})^{\Omega_{\mathcal{F}}}$ and the projector

$$P^{\mathcal{F}} : TT\mathcal{F}|\mathcal{D}^{\mathcal{F}} \longrightarrow \mathcal{C}^{\mathcal{F}}.$$

If $\Gamma : \Omega \times \mathbb{R}_+ \to \mathcal{D}^{\mathcal{F}}$ solves

$$\delta\Gamma = P^{\mathcal{F}} X_{H^i}(\Gamma)\delta W^i$$

then $\rho \circ \tau \circ \Gamma =: \Gamma^{\mathrm{nh}}$ is CONSTRAINED BM in Q.

Theorem

Let $\Pi : TQ = \mathcal{D} \oplus \mathcal{D}^{\perp} \to \mathcal{D}$ be the orthogonal projection and $u = (u^i)$ a local ONF. Then Γ^{nh} is a diffusion with generator

$$\tfrac{1}{2}(\Pi u^i)(\Pi u^i) - \tfrac{1}{2}\Pi\nabla^{\mu}_{\Pi u^i} u^i.$$

Corollary

Let (u^i) be adapted to the composition $\mathcal{D} \oplus \mathcal{D}^{\perp}$. Then it follows that Γ^{nh} is a martingale for the non-holonomic connection ∇^{nh}.

Symmetry reduction

The law of Γ^{nh} is G-invariant and thus Γ^{nh} induces a diffusion Γ^M in $Q/G = M$.

Theorem – Chaplygin case

Let $\mathcal{S} = 0$. Then, for a local ONF (u^a) on (M, μ_0), the generator of Γ^M is of the form

$$\tfrac{1}{2} u^a u^a - \tfrac{1}{2} \nabla^{\mu_0}_{u^a} u^a + \tfrac{1}{2} \nabla^{\mu_0}_{u^a} u^a - \tfrac{1}{2} \nabla^{\text{nh}}_{u^a} u^a = \tfrac{1}{2} \Delta^{\mu_0} + \tfrac{1}{2} b$$

where ∇^{nh} is the non-holonomic connection on M. Further:

$$b = \mu_0^{-1} \beta$$

where β is "preservation-of-measure-form".

Corollary

The reduced diffusion is a martingale in (M, ∇^{nh}). It is BM in (M, μ_0) iff $b = 0$.

Conclusions

Let $\mathcal{S} = 0$ and assume M compact (for simplicity). Then the following are equivalent.

- The deterministic system $X^{\text{nh}}_{\mathcal{H}_0}$ preserves a volume $\mathcal{N}\Omega^m_{TM}$ on TM.
- The non-holonomic diffusion Γ^M is time reversible.
- $\beta = \check{\mu}_0(b)$ is exact, and then $\beta = d(\log \mathcal{N})$.
- The entropy production rate of Γ^M is 0. – purely probabilistic condition.

The *candidate* $\mathcal{N}$, the equilibrium density which exists and is unique for compact M, is characterized by

$$A^*\mathcal{N} = \frac{1}{2}\Delta^\mu \mathcal{N} - \text{div}_{\text{vol}_\mu}(\mathcal{N} b) = 0$$

up to a multiplicative constant. Compactness is satisfied for important examples such as the 2-wheeled carriage, the Chaplygin ball, or the snakeboard ($\mathcal{S} \neq 0$).

Open problems

- What if $\mathcal{S}$ is non-trivial? A bundle picture for non-holonomic systems is needed!

$$\dots \mathcal{D}/G \cong TM \oplus \mathcal{S}/G \dots$$

  and there is a distribution along which an almost symplectic form can be defined.
- Understand some non-Chaplygin examples such as the snakeboard.
- Is there a stochastic Lagrange-d'Alembert principle with constraints?
- Time change is an important topic both in non-holonomic mechanics (*Hamiltonization*) as well as in stochastics. Is there a relation?
- Control of stochastic non-holonomic systems?

References

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- S.H. and T. Ratiu, *The geometry of non-holonomic diffusions*, in preparation.