

SPINNING PARTICLES IN A YANG-MILLS FIELD

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ABSTRACT. Suppose that a Lie group G acts properly on a configuration manifold Q . We study the singular symplectic quotient of T^*Q with respect to the cotangent bundle lifted G -action at an arbitrary coadjoint orbit level \mathcal{O} . In particular, if $Q = Q_{(H)}$ is of single orbit type we show that the symplectic quotient of T^*Q at \mathcal{O} can be constructed through a minimal coupling procedure involving the smaller cotangent bundle T^*Q_H , the symplectic quotient of \mathcal{O} at 0 with respect to the coadjoint H -action, and the diagonal Hamiltonian $N(H)/H$ -action on these symplectic spaces. A prescribed principal connection form on $Q_H \rightarrow Q_H/(N(H)/H)$ then yields a computationally effective way of explicitly realizing the symplectic structure on each stratum of the symplectic quotient of T^*Q . In an example this result is combined with the projection method to produce a stratified Hamiltonian system with very well hidden symmetries.

1. INTRODUCTION

1.A. A brief history of the orbit bundle picture in mechanics. A general discussion of the history of symplectic reduction and its variants can be found in the overview article by Marsden and Weinstein [24]. In particular this article contains historical remarks on the bundle picture in mechanics. The essence of this bundle picture is the following. Let Q be a smooth configuration manifold acted upon in a proper and free fashion by a Lie group G . Then this action can be cotangent lifted to give a Hamiltonian G -action on T^*Q with momentum map $\mu : T^*Q \rightarrow \mathfrak{g}^*$. If \mathcal{O} is a coadjoint orbit in the image of μ then the symplectic quotient $\mu^{-1}(\mathcal{O})/G =: T^*Q//_{\mathcal{O}}G$ is a smooth symplectic manifold. The orbit bundle picture is the observation that, under the additional assumption of a prescribed principal bundle connection (such as the mechanical connection) on $Q \rightarrow Q/G$, one obtains a smooth symplectic fiber bundle

$$\mathcal{O} \hookrightarrow T^*Q//_{\mathcal{O}}G \longrightarrow T^*(Q/G)$$

as well as a description of the reduced symplectic form on the total space of this bundle. ([1, 20, 29, 27, 23]) If the orbit consists of a single point only, i.e., $\mathcal{O} = \{\lambda\}$, this implies that there is a symplectomorphism $T^*Q//_{\lambda}G \cong T^*(Q/G)$ where $T^*(Q/G)$ is equipped with a magnetic symplectic form. In particular, one thus recovers the Abelian version of cotangent bundle reduction as developed earlier in [39, 36].

The bundle picture was motivated by the articles of Sternberg [40] and Weinstein [41] on minimal coupling and Yang-Mills potentials. These articles show how to obtain the equations of motions for a particle in a Yang-Mills field in a symplectic framework.

Date: September 4, 2006.

2000 Mathematics Subject Classification. 53D20, 70S15.

Key words and phrases. Cotangent bundle reduction, singular symplectic reduction, Hamiltonian systems.

This work is supported by Fonds zur Förderung der wissenschaftlichen Forschung (FWF), Projekt P 17108-N04.

If the G -action on Q is assumed to be proper but not necessarily free the symplectic quotient $T^*Q//_{\mathcal{O}}G$ is not a smooth manifold but a stratified symplectic space in the sense of [38, 4, 31]. The case of non-free G -actions has first been considered in Montgomery [26]. In this paper Montgomery uses the point reduction scheme and provides certain conditions under which the point reduced cotangent bundle $T^*Q//_{\lambda}G := \mu^{-1}(\lambda)/G_{\lambda}$ embeds as a subbundle in $T^*(Q^{\lambda}/G_{\lambda})$. Hereby $Q^{\lambda} = \{q \in Q : \lambda \in \text{Ann } \mathfrak{g}_q\}$, and the conditions alluded to are related to the single orbit type assumption of the present paper. Assuming the G -action on Q to be infinitesimally free, Emmrich and Römer [8] have found the symplectic quotient of T^*Q to be an orbifold. More recently, Schmäh [37] has proved a cotangent bundle specific slice theorem at elements $(q, p) \in T^*Q$ whose momentum value $\mu(q, p) = \lambda$ is fully isotropic, i.e., fixed by the $\text{Ad}^*(G)$ -action. Under the same assumption on the momentum value λ , Perlmutter, Rodriguez-Olmos, and Sousa-Diaz [35] were able to describe the geometry of the stratification of the reduced phase space $T^*Q//_{\lambda}G$. Requiring Q to be of single orbit type (that is $Q = Q_{(H)}$), the author found in [13] a generalization of the bundle picture for general momentum values. This point of view was also pursued in [14] to describe the singular Poisson reduced space $(T^*Q)/G$.

1.B. Statement of results. Let G be a Lie group acting properly on a configuration manifold Q . Suppose $Q = Q_{(H)}$ is of single orbit type, that is G_q is conjugate to H within G for all $q \in Q$. Consider the cotangent bundle lifted action by G on T^*Q . This action is proper by assumption and Hamiltonian with momentum map μ . Let \mathcal{O} denote a coadjoint orbit in the image of μ . Clearly the $\text{Ad}^*(H)$ -action on \mathcal{O} is Hamiltonian, albeit in general not free, and we shall denote the singular symplectic quotient of \mathcal{O} by this action at the 0-level by $\mathcal{O}//_0H$. Further, let $Q_H = \{q \in Q : G_q = H\}$ be the symmetry type submanifold of Q on which there is an induced action by $W := N(H)/H$ which is free and proper by construction. Thus there is a diagonal action by W on the product $T^*Q_H \times \mathcal{O}//_0H$ which is free, proper, and Hamiltonian. We show in Theorem 4.6 that $\mu^{-1}(\mathcal{O})/G = T^*Q//_{\mathcal{O}}G$ is isomorphic as a singular symplectic space to the symplectic quotient

$$(T^*Q_H \times \mathcal{O}//_0H)//_0W$$

of $T^*Q_H \times \mathcal{O}//_0H$ with respect to the W -action at the 0-level. Moreover, it is shown that the smooth symplectic strata of $T^*Q//_{\mathcal{O}}G$ can be computed in a similar manner involving T^*Q_H , the smooth symplectic strata of $\mathcal{O}//_0H$, and the induced W -action. If there is a principal bundle connection \mathcal{A} given on $W \hookrightarrow Q_H \rightarrow B := Q_H/W \cong Q_{(H)}/G$ (e.g., the mechanical connection) then this minimal coupling construction yields a singular symplectic fiber bundle

$$\mathcal{O}//_0H \hookrightarrow (T^*Q_H \times \mathcal{O}//_0H)//_0W \cong_{\mathcal{A}} (Q_H \times_B T^*B) \times_W \mathcal{O}//_0H \longrightarrow T^*B$$

and, furthermore, provides an explicit description of the reduced symplectic structure on each of the symplectic strata of $T^*Q//_{\mathcal{O}}G$. This exposition of the reduced symplectic structure is isomorphic to that given in [13]. However, the advantage of the minimal coupling approach is that it is computationally much more effective. For instance, in the example presented in Section 6 it is virtually impossible to obtain the reduced symplectic form by applying the bundle picture construction of [13]. Another point of this proposed construction is that it is specific to singular cotangent bundle reduction. Indeed, if the G -action on Q is free then the result reduces to the shifting trick.

The physical interpretation of the reduced space $(T^*Q_H \times \mathcal{O}//_0H)//_0W$ is that it is the phase space of particles moving on B in the presence of a Yang-Mills field and subject to additional internal variables corresponding to $\mathcal{O}//_0H$. These internal

variables could, for example, be spin variables. This is the case in the Calogero-Moser models considered in [16, 2, 13, 9]. Thus there is a qualitative difference between singular ($H \neq \{e\}$) and regular ($H = \{e\}$) cotangent bundle reduction. In the regular case the gauge group is G and the internal variables correspond to an $\text{Ad}^*(G)$ -orbit, while in the singular case the relation between gauge group W and internal variables $\mathcal{O} //_0 H$ is more intricate.

In Section 5 we make some comments on the general problem of cotangent bundle reduction. This is the problem of understanding the symplectic quotient $T^*Q //_{\mathcal{O}} G$ for general proper G -actions on the configuration space Q . The view of Section 5 is, in principle, that one should first compute the spaces $T^*(Q_{(H)}) //_{\mathcal{O}} G$ via Theorem 4.6 and then proceed by a case to case study to obtain the full symplectic quotient $T^*Q //_{\mathcal{O}} G$.

In Section 6 these ideas are applied to the diagonal action of $G = \text{SO}(5)$ on $Q = S^9 \subset \mathbb{R}^5 \times \mathbb{R}^5$. The stratification of $(T^*S^9)/\text{SO}(5)$ is exhibited according to the remarks in Section 5. Then a special coadjoint orbit level $\mathcal{O} \subset \mathfrak{so}(5)^*$ is fixed. Employing Theorem 4.6 the reduced space $T^*S^9 //_{\mathcal{O}} \text{SO}(5)$ together with the induced symplectic form on each of its strata is computed. Moreover, this reduction process is carried out in the presence of a $\text{SO}(5)$ -invariant Hamiltonian function on T^*S^9 whence we arrive at a stratified Hamiltonian system. In fact, the example is chosen so that there are only two strata and the induced Hamiltonian system on each of these is described shortly. The system on the small stratum corresponds to the motion of a charged particle on a closed disk under the influence of an electromagnetic field. The system on the big stratum is more complicated and a physical interpretation is attempted at the end of Section 6.

2. PRELIMINARIES AND NOTATION

All manifolds to be considered are Hausdorff, paracompact, finite dimensional, and smooth in the C^∞ -sense. Let (M, ω) be a Hamiltonian G -space, i.e., (M, ω) is a symplectic manifold acted upon from the left via symplectomorphisms by a Lie group G such that there is an equivariant momentum map $J : M \rightarrow \mathfrak{g}^*$. Whenever the momentum map is clear from the context we will write $M //_{\lambda} G$ for $J^{-1}(\lambda)/G_{\lambda}$ and $M //_{\mathcal{O}} G$ for $J^{-1}(\mathcal{O})/G$ where \mathcal{O} is a coadjoint orbit. If the action is written as $l : G \times M \rightarrow M$, $(k, x) \mapsto l(k, x) = l_k(x) = l^x(k) = k.x$ we can tangent bundle lift it via $k.(x, v) := (k.x, k.v) := Tl_k.(x, v) = (l_k(x), T_x l_k.v)$ for $(x, v) \in TM$ to an action on TM . As the action consists of transformations by diffeomorphisms it may also be lifted to the cotangent bundle. This is the cotangent lifted action which is defined by $k.(x, p) := (k.x, k.p) := T^*l_k.(x, p) = (k.x, T_{k.x}^* l_{k^{-1}.p})$ where $(x, p) \in T^*M$. Our notation for the fundamental vector field is $\zeta_X(x) := \frac{\partial}{\partial t} |_{t=0} l(\exp(+tX), x) = T_e l^x(X)$ where $X \in \mathfrak{g}$.

If the action by G on M is proper, in the sense that $G \times M \rightarrow M \times M$, $(k, x) \mapsto (x, l(k, x))$ is a proper mapping, then we have the Slice and Tube Theorem at our disposal. An exposition of these facts can be found in Palais and Terng [33], Duistermaat and Kolk [7], or Kawakubo [15], for example. We say M is a proper G -manifold if G is a Lie group acting properly on M .

Let $x \in M$ and $H := G_x$. Then we define $(H) := \{kHk^{-1} : k \in G\}$ to be the G -CONJUGACY CLASS of H , and say that x is of ISOTROPY TYPE (H) . If G is compact we can define a partial ordering on the set of all conjugacy classes (i.e., isotropy types) of the G -action on M as follows. Namely, say $(H) \prec (L)$ if L is conjugate to a subgroup of H .

The submanifold of all points $x \in M$ such that $(G_x) = (H)$ is denoted by $M_{(H)}$, that is,

$$M_{(H)} = \{q \in M : G_q \text{ is conjugate to } H \text{ within } G\},$$

and this is called the ISOTROPY TYPE or ORBIT TYPE submanifold of M of type (H) . Further, let

$$M_H := \{q \in Q : G_q = H\} \text{ and } M^H := \{q \in Q : H \subset G_q\}$$

denote the SYMMETRY TYPE and FIXED POINT submanifold, respectively, of H .

Theorem 2.1. *Suppose G is compact and M is a connected G -manifold. Then the following are true.*

- (1) *There exists a unique maximal isotropy type (H) characterized by requiring that $\dim H = \inf \{\dim G_x : x \in M\}$ and that H has the least number of connected components among all those isotropy subgroups of G which have dimension equal to $\inf \{\dim G_x : x \in M\}$.*
- (2) *If (H) is the maximal isotropy type then $M_{(H)}$ is open and dense in M and $M_{(H)}/G$ is connected.*

Proof. The proof is based on the Palais' Slice Theorem and can be found in, e.g., Palais and Terng [33]. \square

Main assumptions. Throughout the paper we let Q be a connected manifold, and G a connected Lie group which acts properly from the left on Q . We will write the action alternatively as $l(k, q) = l^q(k) = l_k(q) = k.q$ where $k \in G$ and $q \in Q$. We will further assume throughout that $Q_{(H)}$ is non-empty where (H) is a fixed isotropy type of the G -action. Clearly $G.Q_H = Q_{(H)}$, and we have an induced action by G on $Q_{(H)}$ which is of single isotropy type, and we have an induced free action by $W = W(H) := N(H)/H$ on Q_H . Notice, moreover, that right multiplication in the group induces a left action by W on G/H , that is $w.[k] = [kw^{-1}]$. The following description of $Q_{(H)}$ is the key ingredient of the sequel.

Theorem 2.2 (Structure Theorem). *The orbit projection $Q_{(H)} \rightarrow Q_{(H)}/G$ is a smooth fiber bundle with typical fiber G/H . Moreover, it is diffeomorphic to the bundle associated to the principal bundle $Q_H \rightarrow Q_H/W$ with respect to the left action by W on G/H . The diffeomorphism is given by $\kappa : Q_H \times_W G/H \rightarrow Q_{(H)}$, $[(q, kH)]_W \mapsto k.q$.*

Proof. See Duistermaat and Kolk [7]. \square

Remark 2.3. In Palais and Terng [33] the above theorem is proved for proper Fredholm Riemannian G -manifolds. These are Riemannian manifolds M which are modeled on a Hilbert space such that the G -action is proper and isometric, and such that the tangent map at each point of the orbit projection mapping $M \rightarrow M/G$ is a Fredholm map of Hilbert spaces. It should thus be possible to generalize Theorem 4.6 to this setting.

Remark 2.4. Let (M, ω) be a Hamiltonian G -space such that the G -action on M is proper and the momentum map is denoted by J . By [38, 4, 31] the symplectic quotient $J^{-1}(\mathcal{O})/G = M//_{\mathcal{O}}G$ is a stratified symplectic space. This means that the reduced space is a Whitney (B)-stratified space as defined in [25, 7], its strata are smooth symplectic manifolds, and the inclusion mapping of each stratum into $M//_{\mathcal{O}}G$ is a Poisson morphism with respect to the function Poisson structure on $M//_{\mathcal{O}}G$. If J is not equivariant it is for reasons explained in [31] important to consider the stratification of $M//_{\mathcal{O}}G$ by the connected components of the smooth symplectic manifolds $(J^{-1}(\mathcal{O}) \cap M_{(L)})/G$ as opposed to considering the stratification

given by the disconnected pieces $(J^{-1}(\mathcal{O}) \cap M_{(L)})/G$. Here (L) denotes a conjugacy class of some isotropy subgroup of the G -action. However, it is notationally simpler to deal with the stratification given by the disconnected pieces, and since all momentum maps appearing throughout this paper will be equivariant by construction we will regard the disconnected symplectic manifolds $(J^{-1}(\mathcal{O}) \cap M_{(L)})/G$ as the strata of the symplectic quotient. One can, of course, at any point pass to the finer stratification given by the connected components of these strata.

On T^*Q we have a canonical momentum map $\mu : T^*Q \rightarrow \mathfrak{g}^*$ given by $\langle \mu(q, p), X \rangle = \langle p, \zeta_X(q) \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the dual pairing in the appropriate sense, and this is referred to as the cotangent bundle momentum map of the lifted G -action. We will denote the cotangent bundle momentum map on $T^*Q_{(H)}$ by the same symbol μ , and, moreover, the $W = N(H)/H$ -momentum map on T^*Q_H will be denoted by μ as well. This will not cause any confusion since the meaning will be clear from the context.

3. DESCRIPTION OF THE BIG PHASE SPACE AND POISSON REDUCTION

The above Theorem 2.2 is expressed in the following diagram.

$$\begin{array}{ccccc}
 Q_{(H)} & \xleftarrow[\cong]{\kappa} & Q_H \times_W G/H & \xleftarrow{\quad} & Q_H \times G/H & \tag{D1} \\
 \downarrow & & \downarrow & & \downarrow \text{pr}_1 \\
 Q_{(H)}/G & \xleftarrow[\cong]{\quad} & Q_H/W & \xleftarrow{\quad} & Q_H
 \end{array}$$

The point to be exploited in the sequel is that κ lifts to a symplectomorphism of the cotangent bundles $T^*Q_{(H)}$ and $T^*(Q_H \times_W G/H)$ which is equivariant with respect to the lifted G -actions. Using commuting reduction (see Marsden et al. [22]) in order to two times interchange Poisson reduction via left multiplication with two-shot symplectic reduction via right multiplication and [1, Theorem 4.3.3] we thus get

$$\begin{aligned}
 (T^*Q_{(H)})/G &\cong T^*(Q_H \times_W G/H)/G = (T^*Q_H \times (T^*G)//_0H)//_0W/G \\
 &= (T^*Q_H \times G \times_H \text{Ann } \mathfrak{h})//_0W/G = (T^*Q_H \times \text{Ann } \mathfrak{h}/H)//_0W.
 \end{aligned}$$

Hereby, as above, H and W act on G by inversion of right multiplication, and these actions are cotangent lifted to T^*G . Thus the problem of understanding $(T^*Q_{(H)})/G$ reduces to that of understanding $(T^*Q_H \times \text{Ann } \mathfrak{h}/H)//_0W$ which is much easier since, firstly, W acts on T^*Q_H freely and, secondly, the involved spaces are smaller. Moreover, in many examples W is an Abelian and sometimes even finite group. Notice also that we can equip \mathfrak{g} with an H -invariant inner product, and thus identify

$$\mathfrak{w} = \text{Lie}(W) \cong \text{Lie}(N(H))/\mathfrak{h} \cong \text{Lie}(N(H)) \cap \mathfrak{h}^\perp = \text{Fix}(H) \cap \mathfrak{h}^\perp = (\mathfrak{h}^\perp)^H.$$

That is, H acts trivially on \mathfrak{w} . We will assume this identification tacitly for the rest of the paper.

Lemma 3.1. *The momentum map $J_W : T^*Q_H \times \text{Ann } \mathfrak{h}/H \rightarrow \mathfrak{w}^*$ of the W -action is given by*

$$J_W(q, p, [\lambda]) = \mu(q, p) - \lambda|\mathfrak{w}$$

where μ is the cotangent bundle momentum map on T^*Q_H with respect to the W -action.

Proof. Indeed, notice firstly that $\text{Ann } \mathfrak{h}/H = \mathfrak{g}^*//_0H$ is the Poisson reduced space of \mathfrak{g}^* with respect to the Hamiltonian H -action at $0 \in \mathfrak{h}^*$. More precisely, the

H -action is given by $h.\lambda = \text{Ad}^*(h).\lambda = \lambda \circ \text{Ad}(h^{-1})$. Now, the Hamiltonian $N(H)$ -action which is given by the same formula on \mathfrak{g}^* induces an Hamiltonian action on $\text{Ann } \mathfrak{h}/H$. This induced action is the action by $W = N(H)/H$ under consideration. Its momentum map thus computes straightforwardly to be $\text{Ann } \mathfrak{h}/H \rightarrow \mathfrak{w}^*$, $[\lambda] \mapsto -\lambda|_{\mathfrak{w}}$. \square

Let Ver denote the vertical subbundle of TQ_H with respect to the the principal bundle projection $Q_H \rightarrow B$, and Hor^* shall denote the dual horizontal subspace, i.e., the subspace of covectors which annihilate horizontal vectors. Assume there is a principal bundle connection form $\mathcal{A} : TQ_H \rightarrow \mathfrak{w}$ given. Using this connection form we define Hor and Ver^* in the usual way. Since \mathcal{A} reproduces generators of fundamental vectorfields the definition of the cotangent bundle momentum map $\mu : TQ_H \rightarrow \mathfrak{w}^*$ implies that $\text{Hor}_q^* = \mu_q^{-1}(0)$ where $\mu_q := \mu|_{T_q^*Q_H}$, and, further, $\mathcal{A}_q^* : \mathfrak{w}^* \rightarrow \text{Ver}_q^*$ provides an inverse to $\mu_q|_{\text{Ver}_q^*}$. Thus the zero level set of J_W turns out to be

$$\begin{aligned} J_W^{-1}(0) &\cong \{(q, p_0 + \mathcal{A}_q^*(\lambda_0), [\lambda_0 + \lambda_1]_H) : p_0 \in \text{Hor}_q^*, \lambda_0 \in \mathfrak{w}^*, \lambda_1 \in \text{Ann}(\mathfrak{h} + \mathfrak{w})\} \\ &\cong \text{Hor}^* \times \mathfrak{w}^* \times \text{Ann}(\mathfrak{h} + \mathfrak{w})/H. \end{aligned}$$

Therefore, we get the following description of the reduced space $(T^*Q_{(H)})/G$.

Proposition 3.2. *Under the above assumptions $(T^*Q_{(H)})/G$ is isomorphic as a stratified space to $(Q_H \times_B T^*B) \times_W (\mathfrak{w}^* \times \text{Ann}(\mathfrak{h} + \mathfrak{w})/H)$ where $B = Q_H/W \cong Q_{(H)}/G$. Moreover,*

$\text{Ann } \mathfrak{h}/H = \mathfrak{w}^* \times \text{Ann}(\mathfrak{h} + \mathfrak{w})/H \hookrightarrow (Q_H \times_B T^*B) \times_W (\mathfrak{w}^* \times \text{Ann}(\mathfrak{h} + \mathfrak{w})/H) \rightarrow T^*B$
is a singular fiber bundle in the sense of [14].

Remark 3.3. We can use the isomorphism of the above proposition to endow $(Q_H \times_B T^*B) \times_W (\mathfrak{w}^* \times \text{Ann}(\mathfrak{h} + \mathfrak{w})/H)$ with a Poisson structure. The induced Poisson structure on $(T^*Q_{(H)})/G$ was described in Hochgerner and Rainer [14]. More explicitly, notice that the G -equivariant diffeomorphism $\kappa : Q_H \times_W G/H \rightarrow Q_{(H)}$ lifts to induce a G -equivariant diffeomorphism $\kappa_0 : (Q_H \times G \times_H \text{Ann } \mathfrak{h})/W \rightarrow \bigsqcup_{q \in Q_{(H)}} \text{Ann } \mathfrak{g}_q = \text{Ver}^*(Q_{(H)} \rightarrow Q_{(H)}/G)$. Thus we get an induced isomorphism $Q_H \times_W \text{Ann } \mathfrak{h}/H \cong (\bigsqcup_{q \in Q_{(H)}} \text{Ann } \mathfrak{g}_q)/G$ of stratified spaces. Moreover, pulling back commutes with forming associated bundles, that is $(Q_H \times_B T^*B) \times_W (G \times_H \text{Ann } \mathfrak{h}) \cong T^*B \times_B (Q_H \times_W (G \times_H \text{Ann } \mathfrak{h}))$. We therefore get

$$\begin{aligned} (Q_H \times_B T^*B) \times_W (\mathfrak{w}^* \times \text{Ann}(\mathfrak{h} + \mathfrak{w})/H) &\cong \\ &\cong T^*B \times_B (Q_H \times_W \text{Ann } \mathfrak{h}/H) \cong T^*(Q_{(H)}/G) \times_{Q_{(H)}/G} (\bigsqcup_{q \in Q_{(H)}} \text{Ann } \mathfrak{g}_q)/G \end{aligned}$$

where the last space is the Weinstein space description of $(T^*Q_{(H)})/G$ whose Poisson structure is described in [14, Theorem 5.11]. The first isomorphism in the above equation can be seen as a singular instance of the Weinstein picture being equivalent to the Sternberg description. (See Perlmutter and Ratiu [34] for the C^∞ -regular version of these descriptions which includes explicit formulas for the reduced Poisson structures.)

4. THE REDUCED PHASE SPACE

Let $\pi_W : Q_H \rightarrow Q_H/W = B$ and $\rho : Q_H \times G/H \rightarrow Q_H \times_W G/H$ denote the orbit projections.

4.A. The mechanical connection. Assume G acts by isometries on Q with respect to a Riemannian metric g . We denote the restriction of g to the totally geodesic submanifold Q_H by g again. Thus W acts by isometries on (Q_H, g) .

In this setting $\mu : T^*Q_H \rightarrow \mathfrak{w}^*$ yields a natural connection form on the principal bundle $W \hookrightarrow Q_H \rightarrow Q_H/W =: B$ as follows. Define the MOMENT OF INERTIA TENSOR $\mathbb{I} : Q_H \rightarrow \mathfrak{w}^* \otimes \mathfrak{w}^*$ by $\mathbb{I}_q(X, Y) = g_q(\zeta_X(q), \zeta_Y(q))$ where $X, Y \in \mathfrak{w}$. Since the W -action on Q_H is by isometries it is clear that $\mathbb{I}_{w \cdot q}(\text{Ad}(w).X, \text{Ad}(w).Y) = \mathbb{I}_q(X, Y)$ for all $w \in W$, whence \mathbb{I} defines a smooth family of inner products on \mathfrak{w} depending on $q \in Q_H$. The natural connection form to be thus constructed is the MECHANICAL CONNECTION \mathcal{A} defined by

$$\begin{array}{ccc} T_q Q_H & \xrightarrow{\mathcal{A}_q} & \mathfrak{w} \\ g_q \downarrow \cong & & \cong \uparrow \mathbb{I}_q \\ T_q^* Q_H & \xrightarrow{\mu_q} & \mathfrak{w}^* \end{array} \quad (\text{D2})$$

Let $\Phi := \zeta^W \circ \mathcal{A} \in \Omega^1(Q_H; TQ_H)$ be the principal connection associated to the connection form \mathcal{A} .

4.B. The mechanical curvature. Continue to assume that G acts on Q by isometries. The following diagram shows that the generalized mechanical connection A of the bundle $\pi : Q_{(H)} \rightarrow Q_{(H)}/G$ as defined in [13, Section 5.A] is associated to the mechanical connection \mathcal{A} of Subsection 4.A. The term generalized connection is to be understood in the context of Alekseevsky and Michor [3]. Remember from Section 3 that we have chosen an $\text{Ad}(H)$ -invariant inner product on \mathfrak{g} . Thus we obtain a projection $\mathfrak{g} \rightarrow \text{Lie}(N(H))$ and, in turn, a projection $\mathfrak{g}/\mathfrak{h} = \mathfrak{h}^\perp \rightarrow \mathfrak{w} = (\mathfrak{h}^\perp)^H$. Using this projection we define the fibered product $(Q_H \times \mathfrak{w}) \times_{\mathfrak{w}} (G \times_H \mathfrak{h}^\perp)$ which we identify with $Q_H \times \mathfrak{w} \times G \times_H \mathfrak{h}^\perp$.

$$\begin{array}{ccccc} TQ_H \times T(G/H) & \xrightarrow{\Phi \times \text{id}} & \text{Ver}(\pi_W) \times T(G/H) & \xrightarrow[A \times \text{id}]{\cong} & Q_H \times \mathfrak{w} \times G \times_H \mathfrak{h}^\perp \\ T\rho \downarrow & & \downarrow T\rho & & \downarrow \rho_0 \\ T(Q_H \times_W G/H) & \xrightarrow{\tilde{\Phi}} & \text{Ver}(Q_H \times_W G/H \rightarrow B) & \xrightarrow[\cong]{\tilde{A}} & Q_H \times_W (G \times_H \mathfrak{h}^\perp) \\ T\kappa \downarrow \cong & & \downarrow T\kappa & & \cong \downarrow \kappa_0 \\ TQ_{(H)} & \xrightarrow{(\kappa^{-1})^* \tilde{\Phi}} & \text{Ver}(Q_{(H)}) \rightarrow Q_{(H)}/G & \xrightarrow{A} & \bigsqcup_{q \in Q_{(H)}} \mathfrak{g}/\mathfrak{g}_q \end{array}$$

where

$$\begin{aligned} \rho_0 : Q_H \times \mathfrak{w} \times G \times_H \mathfrak{h}^\perp &\longrightarrow Q_H \times_W (G \times_H \mathfrak{h}^\perp), \\ (q, Z, [(g, X)]_H) &\longmapsto [(q, [(g, X)]_H)]_W, \\ \text{and } \kappa_0 : Q_H \times_W (G \times_H \mathfrak{h}^\perp) &\longrightarrow \bigsqcup_{q \in Q_{(H)}} \mathfrak{g}/\mathfrak{g}_q, \\ [(q, [(g, X)]_H)]_W &\longmapsto (g \cdot q, \text{Ad}(g).X + \mathfrak{g}_{g \cdot q}). \end{aligned}$$

In particular, by construction, $\Phi \times \text{id}$ and $(\kappa^{-1})^* \tilde{\Phi} = \zeta^G \circ A$ are $\kappa \circ \rho$ -related. Thus by [19, Theorem 8.15(7)] the same is true for the respective curvatures. That is, the curvatures

$$R_{\Phi \times \text{id}} := \frac{1}{2}[\Phi \times \text{id}, \Phi \times \text{id}] = \frac{1}{2}[\Phi, \Phi] \times 0 = R_\Phi \times 0 = (-\zeta^W \circ \text{Curv}^A) \times 0$$

and

$$R_{\tilde{\Phi}} := \frac{1}{2}[\tilde{\Phi}, \tilde{\Phi}] = -\zeta^G \circ \text{Curv}^A$$

are $\kappa \circ \rho$ -related. The bracket that appears here is the Frölicher-Nijenhuis bracket, and relatedness means that $T(\kappa \circ \rho) \circ (R_{\mathfrak{F}} \times 0) = R_{\mathfrak{F}} \circ (T(\kappa \circ \rho) \oplus T(\kappa \circ \rho))$. For the curvature forms this implies

$$T(\kappa \circ \rho) \circ ((\zeta^W \circ \text{Curv}^A) \times 0) = \zeta^G \circ \text{Curv}^A \circ \oplus^2 T(\kappa \circ \rho).$$

Thus we arrive at the following assertion which, for emphasis, we record as a proposition.

Proposition 4.1. *The generalized mechanical curvature Curv^A is given by the formula*

$$\text{Curv}^A \circ \oplus^2 T(\kappa \circ \rho) = \kappa_0 \circ \rho_0 \circ (\text{Curv}^A \times 0).$$

Therefore, $\text{Curv}^A : \Lambda^2 TQ_{(H)} \rightarrow \mathfrak{g}$ is G -equivariant, and $\text{Ad}(h) \cdot \text{Curv}^A(v, w)(q) = \text{Curv}^A(v, w)(q)$ for all $h \in G_q$. Moreover, Curv^A is horizontal and thus drops to a closed two-form Curv_0^A on $B = Q_H/W$ with values in the adjoint bundle $Q_H \times_W \mathfrak{w}$, and the isomorphisms $B \cong Q_{(H)}/G$, $Q_H \times_W \mathfrak{w} \cong (\bigsqcup_{q \in Q_{(H)}} \text{Fix}(G_q) \cap \mathfrak{g}/\mathfrak{g}_q)/G$ relate Curv_0^A and Curv^A .

Remark 4.2. Notice that this equation greatly facilitates the work one has to do in computing the mechanical curvature Curv^A in examples. Furthermore, it gives a geometrically satisfactory explanation of the otherwise somewhat surprising properties of [14, Proposition 4.1].

Remark 4.3. Of course, Proposition 4.1 is valid for the curvature form Curv^A of any connection form A associated to any principal (not necessarily mechanical) connection form \mathcal{A} on $\pi_W : Q_H \rightarrow B$. However, since in applications we are concerned with mechanical connections only we chose to state it in this way.

4.C. Cotangent bundle reduction via minimal coupling. Combining Propositions 3.2 and 4.1 with the results of [14, Section 6] one is lead to expect that the singular symplectic leaves of the Poisson reduced phase space $(T^*Q_{(H)})/G$ are given by spaces of the form $(Q_H \times_B T^*B) \times_W \mathcal{O} //_0 H$ where \mathcal{O} is a coadjoint orbit of G and $\mathcal{O} //_0 H = (\mathcal{O} \cap \text{Ann } \mathfrak{h})/H$. The adjective singular means that these leaves are actually stratified symplectic spaces, and the smooth symplectic leaves of $(T^*Q_{(H)})/G$ are thus given by the connected components of the smooth symplectic strata of $(Q_H \times_B T^*B) \times_W \mathcal{O} //_0 H$.

For the following, let

$$(\mathcal{O} //_0 H)_{(L_0)H} := (\mathcal{O}_{(L_0)H} \cap \text{Ann } \mathfrak{h})/H$$

where $\mathcal{O}_{(L_0)H}$ denotes the set of $\lambda \in \mathcal{O}$ such that $H_\lambda = G_\lambda \cap H$ is conjugate to $L_0 \subset H$ within H . By virtue of singular symplectic reduction ([38, 31]) the smooth manifold $(\mathcal{O} //_0 H)_{(L_0)H}$ inherits a symplectic form $\Omega_{(L_0)H}^{\mathcal{O}}$ from the (positive) KKS-form on the coadjoint orbit \mathcal{O} . Further, we define

$$(\mathcal{O} //_0 H)_{(L_0)^{N(H)}} := (\mathcal{O}_{(L_0)^{N(H)}} \cap \text{Ann } \mathfrak{h})/H$$

where $(L_0)^{N(H)}$ denotes the conjugacy class of $L_0 \subset H$ in $N(H)$. Recall that $W := N(H)/H$.

Lemma 4.4. *The following are equivalent.*

- (1) For all $n \in N(H)$ there is an $h \in H$ such that $nL_0n^{-1} = hL_0h^{-1}$, that is, $(L_0)^{N(H)} = (L_0)^H$.
- (2) $N_{N(H)}(L_0)/N_H(L_0) = W$.

If these conditions are satisfied then the action by $N(H)$ on \mathcal{O} induces a Hamiltonian action by W on $(\mathcal{O} //_0 H)_{(L_0)H}$.

There is always an induced action by $N_{N(H)}(L_0)/N_H(L_0)$ on $(\mathcal{O} //_0 H)_{(L_0)H}$, and this action is Hamiltonian.

Proof. This follows from the fact that $H_{n,\lambda} = nH_\lambda n^{-1} \subset H$ for all $n \in N(H)$ and $\lambda \in \mathcal{O}$. \square

In view of the previous lemma we will need to consider the space $W.(\mathcal{O} //_0 H)_{(L_0)H}$ which we will refer to as the W -sweep of $(\mathcal{O} //_0 H)_{(L_0)H}$.

Lemma 4.5. *The W -sweep of $(\mathcal{O} //_0 H)_{(L_0)H}$ is a coproduct*

$$\bigsqcup_{[n] \in W} (\mathcal{O} //_0 H)_{(nL_0 n^{-1})H}$$

where $[n] = nH \in N(H)/H = W$, and has the following properties.

- (1) *It is a finite disjoint union of strata of $\mathcal{O} //_0 H$ (possibly with finitely many connected components) all of which are symplectomorphic to $(\mathcal{O} //_0 H)_{(L_0)H}$.*
- (2) *$W.(\mathcal{O} //_0 H)_{(L_0)H} = (\mathcal{O} //_0 H)_{(L_0)N(H)}$*
- (3) *When \mathcal{O} is compact then there is exactly one regular stratum in $\mathcal{O} //_0 H$. It is open, dense and connected, and, furthermore, preserved by the W action.*

Proof. Indeed, as in the proof of Lemma 3.1 we notice that W acts by Poisson morphisms on $\mathcal{O} //_0 H$. It is a general fact (e.g., [38]) that homeomorphisms of singular Poisson spaces which are Poisson are also strata preserving. Therefore, W maps strata onto strata, and it is easy to verify that $w = [n] = nH$ maps $(\mathcal{O} //_0 H)_{(L_0)H}$ symplectomorphically onto $(\mathcal{O} //_0 H)_{(nL_0 n^{-1})H}$. Therefore, the space under consideration is a coproduct of the asserted form, and it is even finite since by compactness of H there are only finitely many strata of $\mathcal{O} //_0 H$. This proves (1).

(2) This assertion is straightforward to verify.

(3) By a result of Kirwan [18] the pre-image of 0 of a proper momentum map is always connected. Since \mathcal{O} is compact it thus follows that $\mathcal{O} \cap \text{Ann } \mathfrak{h}$ is connected. Therefore, by [38] the reduced space has a unique open, dense and connected stratum which is characterized by Theorem 2.1. These characterizing properties are preserved by the W -action – whence the assertion. \square

The following theorem uses the notion of a Hamiltonian fiber bundle which is defined in Subsection 4.E below.

Theorem 4.6 (Cotangent bundle reduction as minimal coupling). *Suppose G is a connected Lie group acting properly on a connected manifold Q , and let (H) be an isotropy type of this action. Assume that \mathcal{O} is a coadjoint orbit contained in the image of the cotangent bundle momentum map $\mu : T^*Q_{(H)} \rightarrow \mathfrak{g}^*$. The following are true.*

- (1) *There is an isomorphism of singular symplectic spaces identifying the symplectic reduced space*

$$(T^*Q_{(H)}) //_{\mathcal{O}} G := \mu^{-1}(\mathcal{O}) \cap (T^*Q_{(H)}) / G$$

and the symplectic reduced space at $0 \in \mathfrak{w}^$ with respect to the free diagonal action by $W = N(H)/H$ on $T^*Q_H \times (\mathcal{O} //_0 H)$.*

- (2) *Let \mathcal{A} be a principal connection form on $\pi_W : Q_H \rightarrow B$. Then \mathcal{A} yields a singular fiber bundle*

$$\mathcal{O} //_0 H \hookrightarrow (T^*Q_H \times \mathcal{O} //_0 H) //_0 W \cong (T^*Q_{(H)}) //_{\mathcal{O}} G \longrightarrow T^*B$$

in the sense of [14]. The transition functions of this bundle take values in W which acts on $\mathcal{O} //_0 H$ by Hamiltonian transformations. Thus the fiber bundle is Hamiltonian.

*Suppose, further, that (L) is an isotropy type of the lifted G -action on $T^*Q_{(H)}$. Then L is conjugate to a subgroup $L_0 \subset H$ within G and the following hold.*

(3) *There is a C^∞ -symplectomorphism*

$$((T^*Q_{(H)})//_{\mathcal{O}G}(L) \cong (T^*Q_H \times W.(O//_0H)_{(L_0)^H})//_0W$$

where $T^*Q_H \times W.(O//_0H)_{(L_0)^H}$ is equipped with the obvious product symplectic form.

(4) *Let \mathcal{A} denote a principal connection form on $Q_H \rightarrow Q_H/W = B$ with curvature form $\text{Curv}^{\mathcal{A}}$. The symplectic form on $T^*Q_H \times W.(O//_0H)_{(L_0)^H}$ restricted to the 0 level set of the momentum map J_W of the diagonal W -action can be realized the minimal coupling form*

$$\text{pr}_1^* \eta^* \Omega^B - \langle \phi \circ \text{pr}_2, (\eta \circ \tau \circ \text{pr}_1)^* \text{Curv}_0^{\mathcal{A}} \rangle - \text{pr}_2^* \Omega_{(L_0)}^{\mathcal{O}}$$

where $\phi : W.(O//_0H)_{(L_0)^H} \rightarrow \mathfrak{w}^*$ is the W -momentum map given by $\phi([\lambda]_H) = \lambda|_{\mathfrak{w}}$. Furthermore, Ω^B is the canonical symplectic form on T^*B , $\tau : T^*B \rightarrow B$, $(\text{pr}_1, \text{pr}_2) : T^*Q_H \times W.(O//_0H)_{(L_0)^H} \rightarrow T^*Q_H \times W.(O//_0H)_{(L_0)^H}$ are the obvious projections, $\eta : T^*Q_H \rightarrow T^*B$ is the projection defined by the connection $\mathcal{A} \in \Omega^1(Q_H; \mathfrak{w})$, and $\text{Curv}_0^{\mathcal{A}}$ is the form on B induced from the basic form $\text{Curv}^{\mathcal{A}}$. This coupling form is horizontal and drops to the induced symplectic form on the reduced space $J_W^{-1}(0)/W = (T^*Q_H \times W.(O//_0H)_{(L_0)^H})//_0W$.

(5) *The symplectomorphism of Item 3 is compatible with the description of [13] in the following sense. Let A be a generalized connection form on $Q_{(H)} \rightarrow Q_{(H)}$, and let $\iota : Q_H \hookrightarrow Q_{(H)}$ denote the inclusion and let $\iota_0 : B = Q_H/W \rightarrow Q_{(H)}/G$ be the induced diffeomorphism. Suppose $\mathcal{A} = \iota^* A$ is the induced principal bundle connection form on $\pi_W : Q_H \rightarrow B$ and $\mathcal{W} \cong_{\mathcal{A}} T^*Q_{(H)}$ is the Weinstein realization of [13, Theorem 5.7]. Then the reduced space $(T^*Q_{(H)})//_{\mathcal{O}G}(L) \cong_{\mathcal{A}} (Q_H \times_B T^*B) \times_W W.(O//_0H)_{(L_0)^H}$ is symplectomorphic to $(\mathcal{W})//_{\mathcal{O}G}(L) = T^*(Q_{(H)}/G) \times_{Q_{(H)}/G} (\bigsqcup_{q \in Q_{(H)}} \mathcal{O} \cap \text{Ann } \mathfrak{g}_q)_{(L)}$ by virtue of the map*

$$[(q; b, \beta; [\lambda]_H)]_W \longmapsto (T^*\iota_0(b, \beta), [(q, \lambda)]_G)$$

where $b = \pi_W(q)$.

Proof. Property (1) follows directly from the observation that

$$\begin{aligned} (T^*Q_{(H)})//_{\mathcal{O}G} &\cong T^*(Q_H \times_W G/H)//_{\mathcal{O}G} \\ &\cong (T^*Q_H \times (T^*G)//_0H)//_0W//_{\mathcal{O}G} = (T^*Q_H \times O//_0H)//_0W \end{aligned}$$

where we use commuting reduction to get the second isomorphism which thus is strata and Poisson structure preserving. The first isomorphism of this equation is stratified and Poisson structure preserving since it is constructed from the lifting of the G -equivariant diffeomorphism $Q_{(H)} \cong Q_H \times_W G/H$. Hereby, as above, H and W act on G by inversion of right multiplication, and these actions are cotangent lifted to T^*G .

(2) Indeed, W acts freely and by Hamiltonian transformations on $T^*Q_H \times O//_0H$. The momentum map of this action is

$$J_W : T^*Q_H \times O//_0H \longrightarrow \mathfrak{w}^*, (q, p, [\lambda]_H) \longmapsto \mu(q, p) - \lambda|_{\mathfrak{w}}$$

which is well-defined and computed by the same reduction-in-stages argument as in Lemma 3.1. Let $\mathcal{A}^* : Q_H \times \mathfrak{w}^* \rightarrow \bigsqcup_{q \in Q_H} T_q^*(W, q) = \text{Ver}^*$, $(q, \lambda) \mapsto \mathcal{A}_q^*(\lambda)$ be the dual of \mathcal{A} . By construction \mathcal{A}_q^* is an inverse to $\mu_q|_{\text{Ver}_q^*}$. Therefore,

$$J_W^{-1}(0) \cong_{\mathcal{A}} \{(q, p_0 + \mathcal{A}_q^*(\lambda|_{\mathfrak{w}}), [\lambda]_H) : p_0 \in \text{Hor}_q^*, [\lambda]_H \in O//_0H\} \cong \text{Hor}^* \times O//_0H.$$

which is an isomorphism of stratified spaces since μ_q clearly is H -equivariant. Thus

$$(T^*Q_H \times O//_0H)//_0W = J_W^{-1}(0)/W \cong_{\mathcal{A}} \text{Hor}^* \times_W O//_0H \longrightarrow \text{Hor}^*/W = T^*B$$

is the fiber bundle over T^*B associated to the Hamiltonian W -action on $O//_0H$.

(3) Writing $Q_{(H)}$ as an associated bundle we see, as above, that $T^*Q_{(H)} \cong (T^*Q_H \times T^*G//_0H)//_0W$ as smooth symplectic manifolds. Since H is normal in $N(H)$ by tautology the Regular Reduction in Stages Theorem ([22, 31]) implies that $T^*Q_{(H)} \cong (T^*Q_H \times T^*G)//_0N$ where $N := N(H)$. Therefore, we can describe the symplectic stratum $(T^*Q_{(H)}//_{\mathcal{O}G})_{(L)} := (\mu^{-1}(\mathcal{O}) \cap (T^*Q_{(H)})_{(L)})/G$ as follows.

$$\begin{aligned}
(T^*Q_{(H)}//_{\mathcal{O}G})_{(L)} &\cong ((T^*Q_H \times T^*G)//_0N//_{\mathcal{O}G})_{(L_0)} \\
&\cong ((T^*Q_H \times \mathcal{O})//_0N)_{(L_0)^N} & (2) \\
&= (J_N^{-1}(0) \cap (T^*Q_H \times \mathcal{O})_{(L_0)^N})/N \\
&= (J_N^{-1}(0) \cap (T^*Q_H \times \mathcal{O}_{(L_0)^N}))/N \\
&= \{(q, p, \lambda) : \lambda|_{\mathfrak{n}} = \mu(q, p) \in \mathfrak{w}^* \subset \text{Ann } \mathfrak{h}\}/N \\
&= \{(q, p, \lambda) : \lambda \in \mathcal{O} \cap \text{Ann } \mathfrak{h}, \lambda|_{\mathfrak{w}} = \mu(q, p)\}/H/W \\
&= \{(q, p, [\lambda]_H) \in J_W^{-1}(0) \subset T^*Q_H \times (\mathcal{O}//_0H)_{(L_0)^N}\}/W \\
&= (T^*Q_H \times W.(\mathcal{O}//_0H)_{(L_0)^H})//_0W & (8)
\end{aligned}$$

where J_N is the N -momentum map on $T^*Q_H \times \mathcal{O}$ given by $J_N(q, p, \lambda) = \mu(q, p) - \lambda|_{\mathfrak{n}}$, \mathfrak{n} is the Lie algebra of N , and the non-obvious identifications are verified as follows. For identification (2) notice that $N \times G$ acts in a Hamiltonian and proper fashion on $T^*Q_H \times T^*G$. A typical isotropy group of this action is of the form

$$(N \times G)_{(q,p,k,\lambda)} = \{(h, khk^{-1}) : h \in H_\lambda\} =: L'$$

It is straightforward to check that $((0, \lambda_0), L')$ (where $\lambda_0 \in \mathcal{O} \cap \text{Ann } \mathfrak{h}$) satisfies the Hamiltonian Stages Hypothesis for non-free actions as formulated in [22, Section 10.4] or in [31, Section 9.5]. Thus we can apply the corresponding reduction theorem as given in these references. The result is identification (2). As stated in Remark 2.4 the (induced) momentum maps appearing in these computations are equivariant whence we are not concerned with the connectedness hypothesis that is made in the general formulation of the Reduction in Stages Theorem. The last identification (8) is a consequence of Lemma 4.5.

Item (4) is an application of the result in Sternberg [40]. See also [41, 11].

Assertions (5). Since both reduced spaces under consideration are symplectic descriptions of the same symplectic quotient they clearly are symplectomorphic. The explicit form of the symplectomorphism follows from Proposition 4.1 and Remark 4.3. (For the formula determining the symplectic structure on $\mathcal{W}//_{\mathcal{O}G}$ see [13, Theorem 5.7].) \square

Remark 4.7. Notice that this theorem allows to compute the reduced cotangent bundle $T^*Q_{(H)}//_{\mathcal{O}G}$ without explicitly knowing the momentum map $\mu : T^*Q_{(H)} \rightarrow \mathfrak{g}^*$.

Remark 4.8. Looking again at the proof of Item 3 in the above theorem one could also directly apply the singular version of the Reduction in Stages Theorem as follows. Namely, N acts in a Hamiltonian fashion on $T^*Q_H \times \mathcal{O}$, a typical isotropy type of this action is $(L_0)^N$, and H is normal in N . One can check again that the pair $(0, L_0)$ satisfies the Hamiltonian Stages Hypothesis. Thus the Reduction in Stages Theorem is applicable and using it we get a symplectomorphism

$$(T^*Q_{(H)}//_{\mathcal{O}G})_{(L)} \cong (T^*Q_H \times (\mathcal{O}//_0H)_{(L_0)^H})//_0 \frac{N_{N(H)}(L_0)}{N_H(L_0)}$$

which thus is an equivalent description of the strata of the reduced space. By Lemma 4.4 these descriptions coincide when $(L_0)^N = (L_0)^H$.

Remark 4.9. Note that Item 3 above is a much sharper statement than Item (2) of [13, Theorem 5.7].

Remark 4.10 (Geometric quantization). According to the above theorem a prescribed connection \mathcal{A} yields a symplectic fibration of $T^*Q_{(H)}/\mathcal{O}G$ over T^*B with fiber $\mathcal{O}/\mathcal{O}H$. Using the quantization in stages procedure outlined in [11, Section 4.1] one could thus try to quantize $T^*Q_{(H)}/\mathcal{O}G$ via quantization of base T^*B and fiber $\mathcal{O}/\mathcal{O}H$. It might be interesting to investigate whether this has any useful consequences for the quantization of the (singular) symplectic quotient $T^*Q/\mathcal{O}G$.

Remark 4.11 (A lot of symmetry). Suppose $L_0 = H$ such that $(\mathcal{O}/\mathcal{O}H)_{(H)^H} = \mathcal{O}_H \cap \text{Ann } \mathfrak{h} = \mathcal{O} \cap \mathfrak{w}^*$ which obviously is invariant under W . Then, the smooth minimal coupling space

$$((T^*Q_{(H)})/\mathcal{O}G)_{(H)} \cong (T^*Q_H \times (\mathcal{O} \cap \mathfrak{w}^*))/\mathcal{O}W \cong_{\mathcal{A}} (Q_H \times_B T^*B) \times_W \mathcal{O} \cap \mathfrak{w}^*$$

is a Hamiltonian fiber bundle over T^*B .

Remark 4.12 (A lot of regularity). Assume \mathcal{O} is compact. By Theorem 2.1 there is a unique maximal conjugacy class of the H -action on \mathcal{O} . Suppose $(L_0)^H$ is this class. Then $(\mathcal{O}/\mathcal{O}H)_{(L_0)^H}$ is -as is noted in Lemma 4.5- invariant under the W -action. Therefore, the smooth minimal coupling space

$$\begin{aligned} ((T^*Q_{(H)})/\mathcal{O}G)_{(L_0)} &\cong (T^*Q_H \times (\mathcal{O}/\mathcal{O}H)_{(L_0)^H})/\mathcal{O}W \\ &\cong_{\mathcal{A}} (Q_H \times_B T^*B) \times_W (\mathcal{O}/\mathcal{O}H)_{(L_0)^H} \end{aligned}$$

is a Hamiltonian fiber bundle over T^*B .

4.D. Hamiltonian reduction. Assume G acts on Q by isometries with respect to a Riemannian structure $\langle \cdot, \cdot \rangle$. Let \mathcal{H} be the free Hamiltonian on T^*Q given by the metric on Q , that is $\mathcal{H}(q, p) = \frac{1}{2}\langle p, p \rangle$. To simplify the notation we use the same symbol $\langle \cdot, \cdot \rangle$ for the metric as well as for the cometric. Then Theorem 4.6 gives a way of computing the reduced Hamiltonian \mathcal{H}_0 on the reduced phase space $(T^*Q_{(H)})/\mathcal{O}G \cong (T^*Q_H \times \mathcal{O}/\mathcal{O}H)/\mathcal{O}W$. Namely, for $q \in Q_H$ let \mathbb{I}_q^G be the non-degenerate pairing on \mathfrak{h}^\perp defined by $\mathfrak{h}^\perp \times \mathfrak{h}^\perp \rightarrow \mathbb{R}$, $(X, Y) \mapsto \langle \zeta_X(q), \zeta_Y(q) \rangle$. Note that this pairing can be extended G -equivariantly to define a tensor $\mathbb{I}^G : Q_{(H)} \rightarrow \bigsqcup_{q \in Q_{(H)}} (\mathfrak{g}/\mathfrak{g}_q)^* \otimes (\mathfrak{g}/\mathfrak{g}_q)^*$. This tensor is used in [13] to define the generalized mechanical curvature A on the bundle $Q_{(H)} \rightarrow Q_{(H)}/G$. Notice further, that the inertia tensor $\mathbb{I} = \mathbb{I}^W : Q_H \rightarrow \mathfrak{w}^* \otimes \mathfrak{w}^*$ defined in Section 3 is just the restriction of \mathbb{I}^G to $\bigsqcup_{q \in Q_H} \mathfrak{w}^* \otimes \mathfrak{w}^*$. Conversely, it is not true that \mathbb{I}^G can be computed from \mathbb{I}^W . However, by virtue of Proposition 4.1 we can obtain the generalized mechanical curvature Curv^A from the simpler form Curv^A .

Regarding the computation of \mathcal{H}_0 let $\lambda \in \text{Ann } \mathfrak{h} = (\mathfrak{g}/\mathfrak{h})^*$, and let $X_q(\lambda)$ denote the vector in \mathfrak{h}^\perp associated to λ via the pairing \mathbb{I}_q^G . Then the reduced Hamiltonian is given by

$$\begin{aligned} (T^*Q_H \times \mathcal{O}/\mathcal{O}H)/\mathcal{O}W &\cong_{\mathcal{A}} (Q_H \times_B T^*B) \times_W \mathcal{O}/\mathcal{O}H \longrightarrow \mathbb{R} \\ [(q; q_0, p_0; [\lambda]_H)]_W &\longmapsto \frac{1}{2}\langle p_0, p_0 \rangle_{q_0} + \frac{1}{2}\mathbb{I}_q^G(X_q(\lambda), X_q(\lambda)). \end{aligned}$$

This follows immediately from the identity $\langle A_q^*(\lambda), A_q^*(\lambda) \rangle = \mathbb{I}_q^G(X_q(\lambda), X_q(\lambda))$ where A_q^* is the point wise dual to A_q determined by the metric and the inertia pairing on \mathfrak{h}^\perp .

As the reduced phase space is a stratified symplectic space we get, via restriction of \mathcal{H}_0 , a Hamiltonian system in the usual sense on each stratum $((T^*Q_{(H)})/\mathcal{O}G)_{(L_0)} \cong (T^*Q_H \times W \cdot (\mathcal{O}/\mathcal{O}H)_{(L_0)^H})/\mathcal{O}W$ where $L_0 \subset H$.

Remark 4.13 (Wong's equations). Suppose β is a G -biinvariant metric on \mathfrak{g} , and let g be a G -invariant Riemannian metric on $Q = Q_{(H)}$ such that $\mathbb{I}_q^G = \beta$, independently of $q \in Q$. The corresponding free Hamiltonian system on T^*Q is

the Kaluza-Klein system of Kerner [17]. In the case that the G -action on Q is free, Montgomery [27] has shown that Hamiltonian reduction of the Kaluza-Klein system yields Wong's equations (Wong [42]). It is further observed in [27] that the reduced Kaluza-Klein system is equivalent to Sternberg's minimal coupling Hamiltonian on $T^*Q//_{\mathcal{O}}G \cong (T^*Q_H \times \mathcal{O}//_0H)//_0W \cong_{\mathcal{A}} T^*B \times_B (Q \times_W \mathcal{O}//_0H) \rightarrow T^*B$, whence it is also equivalent to Weinstein's description of the reduced Hamiltonian system. (See [40, 41] or Subsection 4.E). These results carry over to the singular situation as well. This is roughly seen as follows. Using the Slice Theorem to get a local description of $T^*Q//_{\mathcal{O}}G$ as in [13, Theorem 4.4] one can mimic the computations of [27] to obtain Wong's equations from the Kaluza-Klein Hamiltonian system. By Theorem 4.6(5) the resulting Hamiltonian system is equivalent to the system obtained via Hamiltonian reduction of $(T^*Q_H \times \mathcal{O}//_0H, \text{pr}_1^*\Omega^{Q_H} + \text{pr}_2^*\gamma, \frac{1}{2}g + \frac{1}{2}\beta)$ where $(\text{pr}_1, \text{pr}_2) : T^*Q \times \mathcal{O}//_0H \rightarrow T^*Q \times \mathcal{O}//_0H$ denote the Cartesian projections and γ is the stratified form on $\mathcal{O}//_0H$ which restricts to the induced symplectic form on each of the smooth symplectic strata of $\mathcal{O}//_0H$. The resulting Wong's equations for a curve $(c(t), p(t), \lambda(t))$ on a stratum of $T^*B \times_B (Q_H \times_W \mathcal{O}//_0H) \cong_{\mathcal{A}} T^*Q_{(H)}//_{\mathcal{O}}G$ can be stated as

$$\nabla_{c'} c' = -\hat{g}^{-1}(\langle \rho(\lambda), i_{c'} \text{Curv}_0^A \rangle) \text{ and } D_{c'} \lambda = 0$$

where $D_{c'}$ denotes covariant differentiation along c of sections of the bundle $Q_H \times_W \mathcal{O}//_0H$ with respect to the connection associated to \mathcal{A} . This is well defined on each of the smooth strata of this bundle. (The form of these equations is almost the same as that in Montgomery [28] where also a general discussion of Wong's equations can be found.) Moreover by the above, the reduced Kaluza-Klein Hamiltonian is of the form

$$(Q_H \times_B T^*B) \times_W \mathcal{O}//_0H \longrightarrow \mathbb{R}, [(q; q_0, p_0; [\lambda]_H)]_W \longmapsto \frac{1}{2}\langle p_0, p_0 \rangle_{q_0} + \frac{1}{2}\beta(\lambda, \lambda)$$

where we use the same symbol β for the dual metric on \mathfrak{g}^* . As in the regular case this Hamiltonian differs from Sternberg's minimal coupling Hamiltonian only by a Casimir function. Thus the resulting equations of motion coincide.

Remark 4.14 (Another set of Hamiltonian equations). We continue to assume that G acts on $(Q, \langle \cdot, \cdot \rangle)$ by isometries. Thus W acts on $(Q_H, \langle \cdot, \cdot \rangle)$ by isometries as well, and reduction at 0 with respect to W of the free Hamiltonian system $(T^*Q_H, \Omega^{Q_H}, \frac{1}{2}\langle \cdot, \cdot \rangle)$ yields the free Hamiltonian system on T^*B . Let $(\text{pr}_1, \text{pr}_2) : T^*Q \times \mathcal{O}//_0H \rightarrow T^*Q \times \mathcal{O}//_0H$ denote the Cartesian projections, and reduce the W -invariant system $(T^*Q_H \times \mathcal{O}//_0H, \text{pr}_1^*\Omega^{Q_H} + \text{pr}_2^*\gamma, \text{pr}_1^*\frac{1}{2}\langle \cdot, \cdot \rangle)$ at $0 \in \mathfrak{w}^*$ with respect to the diagonal W -action. The Hamiltonian of this reduced system is given by

$$(T^*Q_H \times \mathcal{O}//_0H)//_0W \cong_{\mathcal{A}} (Q_H \times_B T^*B) \times_W \mathcal{O}//_0H \longrightarrow \mathbb{R} \\ [(q; q_0, p_0; [\lambda]_H)]_W \longmapsto \frac{1}{2}\langle p_0, p_0 \rangle_{q_0} + \frac{1}{2}\mathbb{I}_q^W(X_q(\lambda|\mathfrak{w}), X_q(\lambda|\mathfrak{w})).$$

Comparing this expression to the above it is clear that this describes a Hamiltonian system which is in general different from that obtained by Hamiltonian reduction at \mathcal{O} of $(T^*Q_{(H)}, \Omega, \frac{1}{2}\langle \cdot, \cdot \rangle)$. However, considering the restriction of the reduction of the free system on $T^*Q_{(H)}$ to the stratum $((T^*Q_{(H)})//_{\mathcal{O}}G)_{(H)} \cong (T^*Q_H \times (\mathcal{O} \cap \mathfrak{w}^*))//_0W \cong_{\mathcal{A}} (Q_H \times_B T^*B) \times_W \mathcal{O} \cap \mathfrak{w}^*$ shows that, on this stratum, these a priori different systems coincide. Also, if the G -action on $Q_{(H)}$ is free these systems coincide by virtue of the shifting trick.

4.E. Appendix: Minimal coupling and symplectic fibrations. The purpose of this appendix is to shortly say what we mean by minimal coupling. A detailed exposition of the subject can be found in Guillemin, Lerman, and Sternberg [11] which is also the reference for the subsequent. Let $G \hookrightarrow Q \twoheadrightarrow B$ be a principal fiber bundle equipped with a principal bundle connection form $\mathcal{A} \in \Omega^1(Q; \mathfrak{g})$, and

suppose (F, Ω^F) is a right Hamiltonian G -space with momentum map $J_F : F \rightarrow \mathfrak{g}^*$. In [40] Sternberg has shown how to construct from these data a symplectic form on the so-called Sternberg space

$$(Q \times_B T^*B) \times_G F.$$

Weinstein [41] noticed that this result can be obtained in a more symplectic way through the following universal procedure. Namely, cotangent lift the action by G on Q (which we assume to be a proper left action) to the cotangent bundle T^*Q . Thus we can consider the diagonal action on $T^*Q \times F$ where the action on F is inverted. This action is Hamiltonian with momentum map $J := \mu - J_F$ – where μ is the cotangent bundle momentum map. Thus we can do symplectic reduction to get a new symplectic manifold $J^{-1}(0)/G = (T^*Q \times F)//_0G$. Choosing now a connection \mathcal{A} on $Q \rightarrow B$ yields an explicit symplectomorphism with Sternberg's \mathcal{A} -dependent space, i.e.,

$$(T^*Q \times F)//_0G \cong_{\mathcal{A}} T^*B \times_B (Q \times_G F) \cong (Q \times_B T^*B) \times_G F.$$

This point of view is generally called the Weinstein space picture. Either of the isomorphic ways of constructing a symplectic fiber bundle (definition below) $\chi = \chi(\mathcal{A}) : (T^*Q \times F)//_0G \rightarrow T^*B$ out of the data $G \hookrightarrow Q \twoheadrightarrow B$, \mathcal{A} , and (F, Ω^F) is referred to as MINIMAL COUPLING.

The idea is that one may thus start from a Hamiltonian system $(T^*B, \Omega^B, \mathcal{H})$ and obtain a new system on $(T^*Q \times F)//_0G$, together with its induced symplectic structure, with respect to the Hamiltonian $\chi^*\mathcal{H}$. This way of producing a new system is thought of as adjoining some sort of internal variables (like spin) to the original system on B , and the connection \mathcal{A} is interpreted as the potential of a Yang-Mills field which affects the system on B .

A SYMPLECTIC FIBER BUNDLE is a fiber bundle $F \hookrightarrow X \xrightarrow{\pi} M$ with fiber (F, Ω^F) a symplectic manifold such that the transition functions take values in the group of symplectomorphisms of F . Given such data one may ask whether there is a symplectic form on X such that its restriction to a fiber is the prescribed symplectic form on the fiber? In general there is no such globally defined form on X . However, one can give a partial affirmative answer to this question via the coupling form: A two-form ω on X is called FIBER COMPATIBLE if its restriction to a fiber is the prescribed symplectic form on that fiber. Let $\text{Ver} := \ker T\pi$ and suppose Γ is a connection on $\pi : X \rightarrow M$ such that we also have a horizontal subbundle $\text{Hor} = \text{Hor}(\Gamma)$. The connection Γ is called SYMPLECTIC if the associated parallel transport is fiber-wise symplectomorphic. Given such a connection one can define a fiber compatible form $\omega(\Gamma)$ by declaring $i_v\omega(\Gamma) = 0$ for all horizontal fields v and letting $\omega(\Gamma)$ restrict to the prescribed symplectic form on each fiber. Conversely, given a fiber compatible form ω on X we can define

$$\text{Hor}(\omega) := \{v \in TX : \omega(v, w) = 0 \text{ for all } w \in \text{Ver}\}.$$

If $\text{Hor}(\omega) = \text{Hor}(\Gamma)$ then ω is said to be Γ -COMPATIBLE. This definition does not depend on Γ being symplectic. Clearly, $\omega(\Gamma)$ is Γ -compatible.

There exist strong results ([11, 10]) concerning two-forms on the total space of a symplectic fiber bundle $F \hookrightarrow X \xrightarrow{\pi} M$. In particular it is true that every symplectic fiber bundle has a symplectic connection. Since we will not make use of these results we refrain from stating them explicitly and just refer to [11, 10] which contain a detailed discussion. As a matter of fact, [11, Theorem 1.4.1] provides a way of assuring existence and uniqueness after suitable normalization of Γ -compatible forms for the case that F is compact, connected, and simply connected. This uniquely characterized form is called MINIMAL COUPLING FORM of the symplectic fibration.

Let us return to the minimal coupling construction above. The minimal coupling form of $\text{pr}_1 : T^*Q \times F \rightarrow T^*Q$ is simply $\text{pr}_2^* \Omega^F$. Further, let \mathcal{A} continue to denote the principal bundle connection form on $G \hookrightarrow Q \twoheadrightarrow B$. This gives rise to a symplectic connection $\Gamma = \Gamma(\mathcal{A})$ on $F \hookrightarrow X = (T^*Q \times F) //_0 G \rightarrow T^*B = M$. Concerning the associated minimal coupling form ω_Γ let $\tau : T^*Q \rightarrow Q$ be the footpoint projection, and consider $\text{pr}_2^* \Omega^F - \langle \text{pr}_2, (\tau \circ \text{pr}_1)^* \text{Curv}^{\mathcal{A}} \rangle$. This form restricts to an horizontal and G -invariant object on $J^{-1}(0)$ and drops to ω_Γ via the orbit projection $J^{-1}(0) \twoheadrightarrow J^{-1}(0)/G = X = (T^*Q \times F) //_0 G$. Again, for details we refer to [11].

A HAMILTONIAN FIBER BUNDLE is a symplectic fiber bundle whose transition functions take values in the group of Hamiltonian transformations of the fiber.

Therefore, our interpretation of Theorem 4.6 as exhibiting $(T^*Q_{(H)}) //_{\mathcal{O}} G$ as a minimal coupling space is justified.

5. REMARKS ON THE STRATIFICATION OF COTANGENT BUNDLES

Let (H) denote an isotropy type of the G -action on Q , and (L) be an isotropy type of the cotangent lifted G -action. One of the obvious problems with singular reduction of T^*Q with respect to G is that the foot point projection $\tau : T^*Q \rightarrow Q$ is not stratified, i.e., the preimage of a stratum under τ is not equal to a union of strata of T^*Q . This poses a problem for the Hamiltonian dynamics. Indeed, the Hamiltonian flow of a G -invariant function is easily seen to preserve strata $(T^*Q)_{(L)}$, however, it generally neither preserves $(T^*Q)_{(L)}|_{Q_{(H)}}$ nor $(T^*Q)|_{Q_{(H)}}$. Forcing τ to be stratified by further decomposing strata $(T^*Q)_{(L)}$ into pieces of the form $(T^*Q)_{(L)}|_{Q_{(H)}}$ obviously yields a finer stratification of T^*Q which thus needs to be studied. This finer decomposition of T^*Q was called SECONDARY STRATIFICATION in Perlmutter et al. [35], and we shall adopt this terminology.

Let $\text{Ann } Q_{(H)} \rightarrow Q_{(H)}$ denote the subbundle of $(T^*Q)|_{Q_{(H)}}$ consisting of those covectors which vanish upon insertion of a vector tangent to $Q_{(H)}$. Clearly, we have

$$(T^*Q)_{(L)}|_{Q_{(H)}} = (T^*Q_{(H)} \times_{Q_{(H)}} \text{Ann } Q_{(H)})_{(L)},$$

and note that the momentum map $\mu : T^*Q \rightarrow \mathfrak{g}^*$ vanishes on $\text{Ann } Q_{(H)}$. Therefore, for an orbit \mathcal{O} in the image of μ we have that

$$\mu^{-1}(\mathcal{O})|_{Q_{(H)}} = \mu_{(H)}^{-1}(\mathcal{O}) \times_{Q_{(H)}} \text{Ann } Q_{(H)}$$

where $\mu_{(H)}$ denotes the momentum map of the cotangent lifted G -action on $T^*Q_{(H)}$. Thus one should study the fibration

$$(\text{Ann}_q Q_{(H)})^L \hookrightarrow (\mu_{(H)}^{-1}(\mathcal{O}) \times_{Q_{(H)}} \text{Ann } Q_{(H)})_{(L)}/G \longrightarrow (\mu_{(H)}^{-1}(\mathcal{O}))_{(L)}/G.$$

If $Q_{(H)} = Q_{\text{reg}}$ is the regular stratum which is open dense in Q then $\text{Ann } Q_{(H)}$ is trivial. Thus in this case Theorem 4.6 gives a full answer to the reduction problem, and this is the generic case.

Suppose now that $Q_{(H)} = Q_{\text{reg}}$ is the regular stratum of Q . Similarly as in Proposition 3.2 we see that

$$\begin{aligned} (T^*Q)_{(L)}|_{Q_{(H)}} &= (T^*Q_{(H)})_{(L)} \cong (T^*(Q_H \times_W G/H))_{(L)} \\ &= (T^*Q_H \times G \times_H \text{Ann } \mathfrak{h})_{(L)} //_0 W \\ &\cong (Q_H \times_B T^*B) \times_W (G \times_H \text{Ann } (\mathfrak{h} + \mathfrak{w}) \times \mathfrak{w}^*)_{(L)} \end{aligned}$$

where $B = Q_H/W$. The problem in this case thus reduces to understanding the G -action on $G \times_H \text{Ann } (\mathfrak{h} + \mathfrak{w})$ which is given by $g \cdot [(k, \lambda)] = [(gk, \lambda)]$. Now, $[(k, \lambda)] \in (G \times_H \text{Ann } (\mathfrak{h} + \mathfrak{w}))_{(L)}$ if and only if $k^{-1}G_{[(k, \lambda)]}k = G_{[(e, \lambda)]} =: L_0 \sim L$, and the latter is the case if and only if $\lambda \in (\text{Ann } \mathfrak{h})_{L_0}$ with respect to the H -action on $\text{Ann } \mathfrak{h}$. In particular, $L_0 \subseteq H$.

6. SOME WELL HIDDEN SYMMETRIES

Let the matrix group $G := SO(5)$ act on $Q := S^9 \subset \mathbb{R}^5 \times \mathbb{R}^5 = \mathbb{R}^{10}$ through the diagonal action. With respect to this action Q decomposes into two orbit type strata corresponding to $H_0 := SO(3) \hookrightarrow G$ and $H_1 := SO(4) \hookrightarrow G$. Both embeddings are the standard embeddings into the lower right corner of the matrices in G . Using minus one half of the trace form on $\mathfrak{g} = \mathfrak{so}(5)$ we shall tacitly identify \mathfrak{g} and \mathfrak{g}^* .

6.A. The orbit space $S^9/SO(5)$. We write elements $q \in Q \subset \mathbb{R}^5 \times \mathbb{R}^5$ as $q = (q_1^j, q_2^j)_{j=1}^5$. The subset of regular elements $Q_{(H_0)}$ is the set of $(q_1, q_2) \in Q$ such that q_1 and q_2 are linearly independent. In view of the isomorphism $Q_{(H_0)}/G \cong Q_{H_0}/W_0$ where $W_0 := N(H_0)/H_0$ (and Proposition 3.2) we need to identify Q_{H_0} and W_0 . With the embedding $H_0 = SO(3) \hookrightarrow G$ into the lower right corner we have

$$Q_{H_0} = \{((a, b, 0, 0, 0)^t, (\alpha, \beta, 0, 0, 0)^t) : \\ (a, b)^t \text{ and } (\alpha, \beta)^t \text{ are linearly independent and } a^2 + b^2 + \alpha^2 + \beta^2 = 1\}$$

where $(\)^t$ denotes transpose. We embed $Q_{H_0} \hookrightarrow S^3$ in the obvious way. Moreover,

$$W_0 = S^1 \times \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} =: S^1 \times \Delta$$

acts on $Q_{H_0} \subset S^3$ in the standard fashion. Thus the orbit projection $\eta_0 : Q_{H_0} \twoheadrightarrow Q_{H_0}/S^1$ is just a restriction of the Hopf map to an open dense subset $Q_{H_0} \subset S^3$. In order to explicitly describe $B_0 = Q_{H_0}/W_0$ it is convenient to write the Hopf map η from S^3 onto the sphere $S^2(\frac{1}{2})$ of radius $\frac{1}{2}$ as

$$\eta : \mathbb{R}^4 = \mathbb{C}^2 \supset S^3 \longrightarrow S^2(\frac{1}{2}) \\ (q_1, q_2) = (a + ib, \alpha + i\beta) \longmapsto (\frac{1}{2}(|q_1|^2 - |q_2|^2), \Re(q_1 \overline{q_2}), \Im(q_1 \overline{q_2}))^t.$$

Now it is straightforward to see that $\eta_0 = \eta|_{Q_{H_0}}$ takes Q_{H_0} onto the open subset $\{(x, y, z)^t \in S^2(\frac{1}{2}) : z \neq 0\}$, and the Δ -action factors through η_0 to an action by $\{\pm 1\}$ in the z -direction. Therefore, $B_0 = Q_{H_0}/W_0 = \{(x, y, z)^t \in S^2(\frac{1}{2}) : z > 0\}$, and this is the regular stratum of the orbit space $B := Q/G$.

Similarly, to get the other stratum of B notice that

$$Q_{H_1} = \{((a, 0, 0, 0, 0)^t, (\alpha, 0, 0, 0, 0)^t) : a^2 + \alpha^2 = 1\}$$

and $W_1 := N(H_1)/H_1$ is trivial. Thus $Q_{H_1}/W_1 \cong S^1$, and via η we can send this S^1 diffeomorphically onto the equator in the x - y -plane in $S^2(\frac{1}{2})$. Therefore, $B = \{(x, y, z)^t \in S^2(\frac{1}{2}) : z \geq 0\} = B_0 \sqcup B_1$ is stratified into northern hemisphere plus equator.

6.B. The reduced phase space $T^*S^9/SO(5)$. Now we can invoke Proposition 3.2 to compute $(T^*Q_{(H_0)})/G \cong (Q_{H_0} \times_{B_0} T^*B_0) \times_{W_0} (\mathfrak{w}_0^* \times \text{Ann}(\mathfrak{h}_0 + \mathfrak{w}_0)/H_0)$. Indeed, W_0 acts trivially on $\mathfrak{w}_0 = \mathbb{R}$ whence the reduced phase space is an associated bundle (in a singular sense – see [14]) of the type

$$\mathbb{R} \times \text{Ann}(\mathfrak{h}_0 + \mathfrak{w}_0)/H_0 \cong \mathbb{R} \times \mathbb{R}^3 \times_{H_0} \mathbb{R}^3 \hookrightarrow \\ \hookrightarrow (T^*Q_{(H_0)})/G \cong (Q_{H_0} \times_{B_0} T^*B_0) \times_{W_0} (\mathbb{R} \times \mathbb{R}^3 \times_{H_0} \mathbb{R}^3) \longrightarrow T^*B_0$$

where we use the negative of the trace form on \mathfrak{g} to identify $\text{Ann}(\mathfrak{h}_0 + \mathfrak{w}_0) \subset \mathfrak{g}^*$ and $\mathfrak{h}_0^\perp \cap \mathfrak{w}_0^\perp \subset \mathfrak{g}$ as well as the H_0 -equivariant linear isomorphism

$$\mathfrak{h}_0^\perp \cap \mathfrak{w}_0^\perp \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3, (x_{ij})_{ij} \longmapsto ((x_{k1})_{k=3}^5, (x_{k2})_{k=3}^5) = (v, w)$$

where $H_0 = SO(3)$ acts on $\mathbb{R}^3 \times \mathbb{R}^3$ in the standard diagonal way.

According to the remarks in Section 5 we also want to consider the regular stratum $(T^*Q)_{reg} = (T^*Q)_{(L_0)}$ as well as its secondary strata $(T^*Q)_{(L_0)}|_{Q_{(H_0)}}$ and $(T^*Q)_{(L_0)}|_{Q_{(H_1)}}$. It is easy to see that $L_0 = \{1\}$ and that $(T^*Q)_{(L_0)}|_{Q_{(H_1)}}$ is empty. Furthermore, $(G \times_H \text{Ann}(\mathfrak{h}_0 + \mathfrak{w}_0))_{(L_0)} \cong (G \times_H (\mathbb{R}^3 \times \mathbb{R}^3))_{\{1\}}$ consists of those elements $[(k, v, w)]$ for which v and w are linearly independent, that is $(v, w) \in (\mathbb{R}^3 \times \mathbb{R}^3)_{\{1\}}$. Therefore, the regular part of the reduced phase space $(T^*Q)/G$ is

$$\begin{aligned} \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}_{>0} &\hookrightarrow \\ &\hookrightarrow (T^*Q_{(H_0)})_{\{1\}}/G \cong (Q_{H_0} \times_{B_0} T^*B_0) \times_{W_0} (\mathbb{R} \times \mathbb{R}^3 \times_{H_0} \mathbb{R}^3)_{\{1\}} \longrightarrow T^*B_0 \end{aligned}$$

which is a smooth Poisson manifold invariant under the induced Hamiltonian mechanics. (Since $(T^*Q)_{(L_0)}|_{Q_{(H_1)}} = \emptyset$.) Invariance of this stratum is remarkable since this is not a general feature of secondary strata.

Below, when we turn to explicitly describing the induced Poisson structure we will see that this is the phase space of a charged particle with spin moving on B_0 . The charge is governed by the \mathbb{R} -factor while the spin is described by the whole $\mathbb{R} \times (\mathbb{R}^3 \times \mathbb{R}^3)_{\{1\}}/H_0$ -factor.

The next isotropy type in the hierarchy of isotropy types of the G action on T^*Q is $(L_1) = (SO(2))$, and we consider $S^1 = SO(2) = L_1$ as embedded into the lower right corner of G . Now, for an element $x = [(k, v, w)] \in G \times_{H_0} (\mathbb{R}^3 \times \mathbb{R}^3) \cong G \times_{H_0} \text{Ann}(\mathfrak{h}_0 + \mathfrak{w}_0)$ the isotropy group G_x with respect to the induced G -action satisfies $G_x = k(H_0)_{(v,w)}k^{-1}$. Therefore,

$$\begin{aligned} \mathbb{R} \times (\mathbb{R}^3 \times \mathbb{R}^3)_{(S^1)}/H_0 &\cong \mathbb{R} \times C^0(S^1) \hookrightarrow \\ &\hookrightarrow (T^*Q_{(H_0)})_{L_1}/G \cong (Q_{H_0} \times_{B_0} T^*B_0) \times_{W_0} (\mathbb{R} \times \mathbb{R}^3 \times_{H_0} \mathbb{R}^3)_{(S^1)} \longrightarrow T^*B_0 \end{aligned}$$

where $C^0(S^1) \cong ((\mathbb{R} \times \mathbb{R}) \setminus (0, 0))/\{\pm 1\}$ denotes the cone over S^1 without the cone point. This secondary stratum is, however, not invariant under the Hamiltonian dynamics. This means that particles with spin of $C^0(S^1)$ -type may travel from B_0 to B_1 while this is not possible for particles with $(\mathbb{R}^3 \times \mathbb{R}^3)_{\{1\}}/H_0$ spin type.

Considering the isotropy type $(L_2) = (SO(3))$ implies

$$(T^*Q_{(H_0)})_{L_2}/G \cong T^*B_0 \times \mathbb{R} \times \{0\}$$

which, too, is not invariant under the reduced dynamics. The most singular element in the isotropy lattice of T^*Q is $(L_3) = (SO(4))$. However, $(T^*Q)_{(L_3)}|_{(H_0)} = \emptyset$, and we have thus described the secondary stratification of $(T^*Q_{(H_0)})/G$.

6.C. Curvature, Poisson structure, and symplectic leaves. By Section 4, in order to understand the Poisson and symplectic structures of the reduced space $(T^*Q_{H_0})/G$ we have to compute the mechanical connection $\text{Curv}^{\mathcal{A}}$. Thus we first compute the inertia tensor $\mathbb{I} : Q_{H_0} = S_0^3 \rightarrow \mathfrak{w}^* \otimes \mathfrak{w}^*$. That is, for $iX, iY \in i\mathbb{R} = \mathfrak{w}$ and $q \in S_0^3$,

$$\mathbb{I}_q(X, Y) = \langle \frac{\partial}{\partial t} |_0 e^{itX} q, \frac{\partial}{\partial t} |_0 e^{itY} q \rangle = -XY i^2 \langle q, q \rangle = XY$$

which is independent of q .¹ Thus we can identify $\mathbb{R} \cong i\mathbb{R} \cong i\mathbb{R}^* \cong \mathbb{R}^*$. Therefore, by diagram (D2),

$$\langle \mathcal{A}(q, v), X \rangle = \langle v, \zeta_X(q) \rangle = \langle v, iq \rangle X$$

where $(q, v) \in TS_0^3$. That is $\mathcal{A}(q, v) = \langle v, iq \rangle$. To better understand this we now use the quaternionic representation of the Hopf map $\eta : S^3 \rightarrow S^2$. I.e., $q =$

¹Not knowing Proposition 4.1 one would have to compute the generalized mechanical connection from the inertia tensor \mathbb{I}^G defined by $\mathbb{I}_q^G(X, Y) = \langle \zeta_X(q), \zeta_Y(q) \rangle$ where $q \in Q_{(H_0)}$ and $X, Y \in \mathfrak{g}$. The resulting equations blow up horribly.

$a + ib + j\alpha + k\beta$, $v = a' + ib' + j\alpha' + k\beta'$, and using the orthonormal frame $\xi_1(q) = iq$, $\xi_2(q) = jq$, $\xi_3(q) = kq$ we trivialize $TS_0^3 = S_0^3 \times \mathfrak{so}(3)_-$. (Here, $\mathfrak{so}(3)_-$ denotes the Lie algebra of right invariant vector fields on $SO(3)$.) The frame vectors ξ_1, ξ_2, ξ_3 and their dual covectors ξ^1, ξ^2, ξ^3 enjoy the relations

$$\begin{aligned} [\xi_2, \xi_3] &= -2\xi_1, [\xi_3, \xi_1] = -2\xi_2, [\xi_1, \xi_2] = -2\xi_3, \\ \text{and } 2\xi^2 \wedge \xi^3 &= d\xi^1, 2\xi^3 \wedge \xi^1 = d\xi^2, 2\xi^1 \wedge \xi^2 = d\xi^3. \end{aligned}$$

In terms of these we get $\mathcal{A} = \xi^1$, and therefore $\text{Curv}^{\mathcal{A}} = d\mathcal{A} = 2\xi^2 \wedge \xi^3$. Since $\eta_0 : S_0^3 \rightarrow B_0$ is a Riemannian submersion the volume form ν -which is the standard one induced from \mathbb{R}^3 - pulls back to $\eta_0^*\nu = \xi^2 \wedge \xi^3$. In other words the mechanical curvature $\text{Curv}^{\mathcal{A}}$ drops via η_0 to $\text{Curv}_0^{\mathcal{A}} = 2\nu$.

An immediate and easily visible consequence is the following. Assume $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$ is such that $\mathcal{O} //_0 H_0 = \{\text{point}\}$ where \mathcal{O} is the adjoint G -orbit through λ . Then $(T^*Q_{H_0})/G = T^*B_0$ is a magnetic cotangent bundle equipped with the symplectic structure $\Omega^{B_0} - 2\langle \lambda, \tau^*\nu \rangle$ where Ω^{B_0} is the canonical structure on T^*B_0 and $\tau : T^*B_0 \rightarrow B_0$ is the projection. This follows from Theorem 4.6.

6.D. Particular cases. As an interesting and representative particular case let $s \neq 0$ and consider the 5×5 matrix $\lambda \in \mathfrak{so}(5) \cong \mathfrak{so}(5)^*$ which has

$$\begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix}$$

in the top left corner and zeros elsewhere. That is, we are starting from points of the form

$$(q_0, p_0, \lambda) \in (T^*Q_{(H_0)})_{L_2}/G \cong T^*B_0 \times \mathbb{R} \times \{0\}$$

and want to compute the singular symplectic leaves passing through these. Therefore, by Theorem 4.6 we are to be concerned with the singular symplectic space $\mathcal{O} //_0 SO(3) = \mathcal{O} \cap \mathfrak{h}_0^\perp / H_0$ where $\mathcal{O} \cong SO(5)/(S^1 \times SO(3))$ is the G -orbit through λ . Now, λ can as well be written as $\lambda = se_1 \wedge e_2$ where e_i is to be the standard basis vector of \mathbb{R}^5 with 1 in the i -th position and zeros elsewhere. Let A, B, C, D, E denote an arbitrary positively ordered orthonormal basis of \mathbb{R}^5 so that $k = (A|B|C|D|E)$ is an arbitrary element of $SO(5) = G$. Then

$$k\lambda k^{-1} = s(ke_1) \wedge (ke_2) = sA \wedge B$$

and the condition for $k\lambda k^{-1}$ to be in \mathfrak{h}_0^\perp translates to

$$\begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix} \times \begin{pmatrix} b_3 \\ b_4 \\ b_5 \end{pmatrix} = 0,$$

i.e., the vectors formed by the latter 3 components of A and B are to be linearly dependent. Via the H_0 action we can thus bring A and B to the normal form $A = (a_1, a_2, a_3, 0, 0)^t$ and $B = (b_1, b_2, b_3, 0, 0)^t$. It should be noted that this induces an action of $\{\pm 1\} = N_{H_0}(S^1)/S^1$ in the third component which eliminates the remaining freedom. In other words, the 0-level set of the H_0 -momentum map is a smooth manifold of the form

$$\mathcal{O} \cap \mathfrak{h}_0^\perp \cong (\mathcal{O} \cap \mathfrak{h}_0^\perp)^{S^1} \times_{\{\pm 1\}} H_0/S^1 = S^2(s) \times_{\{\pm 1\}} H_0/S^1.$$

Since the orthonormality conditions on A and B are preserved this means that $A \wedge B \in SO(3)/S^1$. The quotient $\mathcal{O} //_0 H_0$ can thus be described as

$$(a_1, a_2, a_3)^t \times (b_1, b_2, b_3)^t = (x_1, x_2, x_3)^t \in V(2, 3)/S^1 = Gr(2, 3) = S^2$$

modulo the remaining ± 1 -action in the third component which induces a ± 1 -action on $(x_1, x_2) = (a_2b_3 - b_2a_3, -a_1b_3 + b_1a_3)$. I.e., $\mathcal{O} //_0 H_0$ is the following

stratified space: the regular stratum consists of $[(x_1, x_2, x_3)^t]_{\sim} \in S_0^2(s)/\sim$ where $(x_1, x_2, x_3) \sim (-x_1, -x_2, x_3)$ and $(x_1, x_2) \neq (0, 0)$; and the singular stratum is $\{(0, 0, s), (0, 0, -s)\}$. In fact, the singularities occurring here are only orbifold singularities.

This decomposition of $\mathcal{O}/\!/_0H_0$ induces the symplectic (Sjamaar-Lerman) stratification on the singular symplectic product space $T^*Q_{H_0} \times \mathcal{O}/\!/_0H_0$ – which by Theorem 4.6 can be further reduced to $(T^*Q_{(H_0)})/\!/_\mathcal{O}G \cong (T^*Q_{H_0} \times \mathcal{O}/\!/_0H_0)/\!/_0W_0$. Its singular (i.e., lower dimensional) strata $T^*Q_{H_0} \times \{\pm s\}$ reduce via W to T^*B_0 with the magnetic symplectic form $\Omega^{B_0} - 2\langle \pm s, \tau^*\nu \rangle$. For the regular stratum let

$$\pi_1 : T^*Q_{H_0} \times S_0^2(s)/\sim \rightarrow T^*Q_{H_0} \text{ and } \pi_2 : T^*Q_{H_0} \times S_0^2(s)/\sim \rightarrow S_0^2(s)/\sim$$

denote the Cartesian projections, let $\eta : T^*Q_{H_0} \rightarrow T^*B_0$ denote the projection associated to the horizontal lifting map with respect to $\mathcal{A} : TQ_{H_0} \rightarrow \mathfrak{w}_0$, and let $\text{pr}_0 : S_0^2(s)/\sim \rightarrow \mathfrak{w}_0$, $[(x_1, x_2, x_3)^t]_{\sim} \mapsto x_3$ denote the induced momentum map of the induced Hamiltonian action by W_0 on $S_0^2(s)/\sim$. (Again, see Theorem 4.6.) Then the reduced symplectic form on the regular piece $(T^*Q_{H_0} \times S_0^2(s)/\sim)/\!/_0W_0$ is the one induced by the basic form

$$\pi_1^*\eta^*\Omega^{B_0} - 2\langle \text{pr}_0 \circ \pi_2, (\pi_{W_0} \circ \tau \circ \pi_1)^*\nu \rangle + \pi_2^*\gamma \quad (\text{MCF})$$

on $J_{W_0}^{-1}(0) \subset T^*Q_{H_0} \times S_0^2(s)/\sim$. Here, γ is the reduced form on the surface $S_0^2(s)/\sim = (\mathcal{O}/\!/_0H_0)_{\text{regular}}$, and $J_{W_0} = \mu_{Q_{H_0}} - \text{pr}_0$ is the momentum map on $T^*Q_{H_0} \times S_0^2(s)/\sim$ with respect to the diagonal Hamiltonian W_0 -action. Notice that the middle term in this formula makes the spin variables interact with the magnetic field 2ν .

6.E. Hamiltonian reduction. Let us carry out the reduction at level λ as in Subsection 6.D once again but this time in the presence of a Hamiltonian \mathcal{H} . The obvious G -invariant Hamiltonian to look at is the one associated to the round metric on S^9 , i.e., $\mathcal{H} : T^*S^9 \rightarrow \mathbb{R}$, $(q, p) \mapsto \frac{1}{2}\langle p, p \rangle$. Use this metric to identify $TS^9 = T^*S^9$, and to split $TQ_{(H_0)} = \text{Hor} \oplus \text{Ver}$ into horizontal and vertical parts with respect to $Q_{(H_0)} \rightarrow Q_{(H_0)}/G = B_0$. The generalized mechanical connection form A thus induces a collection of point-wise isomorphisms $A_q : \text{Ver}_q \rightarrow \mathfrak{g}_q^\perp$. (See Section 4 and [13].)

By the remark about Hamiltonian reduction in Section 4.C the reduced Hamiltonian function \mathcal{H}_0 is given by

$$\begin{aligned} \mathcal{H}_0 : (Q_{H_0} \times_{B_0} T^*B_0) \times_{W_0} \mathcal{O}/\!/_0H_0 &\longrightarrow \mathbb{R}, \\ [(q; q_0, p_0; [X]_{H_0})]_{W_0} &\longmapsto \frac{1}{2}\langle p_0, p_0 \rangle_{q_0} + \langle A_q^*(X), A_q^*(X) \rangle_q. \end{aligned}$$

To compute the last term of this expression it suffices, by H_0 -invariance, to consider $X \in (\mathcal{O} \cap \mathfrak{h}_0^\perp)^{S^1}$, which by Subsection 6.D is of the form

$$X = \begin{pmatrix} 0 & -z & y & 0 & 0 \\ z & 0 & -x & 0 & 0 \\ -y & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with $x^2 + y^2 + z^2 = s^2$. Let \mathbb{I}^G be the inertia tensor of the G -action on $Q_{(H_0)}$ as in the remark referred to above. That is $\mathbb{I}_q^G(Z, Y) = \langle \zeta_Z(q), \zeta_Y(q) \rangle$ for $q \in Q_{(H_0)}$ and $Z, Y \in \mathfrak{g}/\mathfrak{g}_q \cong \mathfrak{g}_q^\perp$. For $q \in Q_{H_0}$ this thus induces an isomorphism

$$\check{\mathbb{I}}_q^G : \mathfrak{h}^\perp \rightarrow \text{Ann } \mathfrak{h} \cong \mathfrak{h}^\perp.$$

With the metric identifications $TQ_{(H_0)} \cong T^*Q_{(H_0)}$ and $\mathfrak{g} \cong \mathfrak{g}^*$ we are using this implies that the dual to the mechanical connection is given by

$$A_{q_1}^*(X_1) = \zeta_Z(q_1) \text{ where } Z = (\check{\mathbb{I}}_{q_1}^G)^{-1}(X_1)$$

with $q_1 \in Q_{(H_0)}$ and $X_1 \in \mathfrak{g}_{q_1}^\perp$. Let $q = ((a, b, 0, 0, 0)^t, (\alpha, \beta, 0, 0, 0)^t) \in Q_{H_0}$ and X as above. Then it is straightforward to verify that

$$Z = (\check{\mathbb{I}}_q^G)^{-1}(X) = \begin{pmatrix} 0 & -z & \frac{\varphi}{\Delta} & 0 & 0 \\ z & 0 & -\frac{\psi}{\Delta} & 0 & 0 \\ -\frac{\varphi}{\Delta} & \frac{\psi}{\Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\begin{aligned} \Delta &= AB - \varepsilon^2 = (a\beta - b\alpha)^2, \varphi = \varepsilon x + By, \psi = Ax + \varepsilon y, \\ \varepsilon &= ab + \alpha\beta, A = a^2 + b^2, B = \alpha^2 + \beta^2. \end{aligned}$$

Note that, since $q \in Q_{H_0}$, $\Delta \neq 0$ and $A + B = 1$. By slight abuse of notation we identify $((a, b, 0, 0, 0)^t, (\alpha, \beta, 0, 0, 0)^t) = q = ((a, b)^t, (\alpha, \beta)^t) \in S_0^3$ from now on. Therefore, we find

$$\begin{aligned} V(q, X) &:= \frac{1}{2} \langle A_q^*(X), A_q^*(X) \rangle = \frac{1}{2} \langle Z^t Z \cdot q, q \rangle = \frac{1}{2\Delta^2} \left\langle \begin{pmatrix} z\Delta^2 + \varphi^2 & -\varphi\psi \\ -\varphi\psi & z\Delta^2 + \psi^2 \end{pmatrix} \cdot q, q \right\rangle \\ &= \frac{1}{2\Delta} \left\langle \begin{pmatrix} z^2\Delta + x^2 & xy \\ xy & z^2\Delta + y^2 \end{pmatrix} \cdot q, q \right\rangle \end{aligned}$$

Note that $V(q, X) = V([(q, [X]_\sim)]_{W_0})$ which is compatible with the necessary requirement that V drops to a function on the reduced phase space

$$\begin{aligned} T^*Q_{(H_0)} //_{\mathcal{O}} G &= (T^*Q_{H_0} \times \mathcal{O} //_0 H_0) //_0 W_0 \\ &\cong_{\mathcal{A}} (Q_{H_0} \times_{B_0} T^*B_0) \times_{W_0} \mathcal{O} //_0 H_0 \cong T^*B_0 \times_{B_0} (Q_{H_0} \times_{W_0} \mathcal{O} //_0 H_0). \end{aligned}$$

Thus we have computed the reduced Hamiltonian $\mathcal{H}_0 : T^*Q_{(H_0)} //_{\mathcal{O}} G \rightarrow \mathbb{R}$. Since we are dealing with a stratified space we effectively get two Hamiltonian systems. Indeed, the system corresponding to the small stratum where $(x, y, z) = (0, 0, \pm s)$ is

$$\left(T^*B_0, \Omega^{B_0} \mp s2\tau^*\nu, \mathcal{H}_0 = \frac{1}{2} \langle p_0, p_0 \rangle + V(q_0, (0, 0, \pm s)) = \frac{1}{2} \langle p_0, p_0 \rangle + \frac{1}{2} s^2 \right),$$

and its flow equations are the Lorentz equations describing the motion of a charged particle with charge $\pm s$ on B_0 under the influence of the magnetic field 2ν . (In fact, the analysis shows that we can extend the reduced system on B_0 to the reduced system on the orbifold $B = B_0 \sqcup B_1 = Q/G$. I.e., we obtain the reduced system on the full primary or Sjamaar-Lerman stratum $(T^*Q //_{\mathcal{O}} G)_{(H_0)}$ in this way.) Indeed, this can be seen as follows. Let g denote the induced metric on $S^2(\frac{1}{2})$ and J the standard complex structure such that $\hat{g} \circ J = \hat{\nu}$. By Remark 4.13 (or direct computation) the Hamiltonian equations associated to \mathcal{H}_0 (lifted to $T^*S^2(\frac{1}{2})$) yield Lorentz equations for a curve $c(t)$ of the type $\nabla_{c'} c' = \hat{g}^{-1}(\pm s i_{c'} \text{Curv}_0^A) = \pm 2s J(c')$, whence the curves $c(t)$ are small circles of constant geodesic curvature $\langle Jc', \nabla_{c'} c' \rangle = \pm 2s$. Projecting these to $B = B_0 \sqcup B_1$ via the ± 1 action in the z -direction one obtains the solution curves. In particular these solutions can leave B_0 in finite time to be reflected at B_1 back into B_0 in a Snell's law manner.

Emphasizing the point of view of minimal coupling this means that the free Hamiltonian system on T^*B_0 with gauge group W_0 is coupled with $\{(0, 0, \pm s)\} \subset \mathfrak{w}^*$. Thus the particles are equipped with a charge $\pm s$ and are accordingly deflected from their geodesic paths.

The reduced Hamiltonian system on the regular stratum is given by regular reduction at 0 with respect to W_0 of

$$\left(T^*Q_{H_0} \times S_0^2(s)/\sim, \pi_1^*\Omega^{Q_{H_0}} + \pi_2^*\gamma, \mathcal{H}_0 = \frac{1}{2}\langle p_0, p_0 \rangle + V(q, [X]_\sim) \right)$$

where $[X]_\sim \in (\mathcal{O} //_0 H_0)_{\text{reg}} = S_0^2(s)/\sim$. The corresponding reduced symplectic structure is described by the minimal coupling form (MCF) in Subsection 6.D. Let

$$K[X]_\sim := \frac{1}{\Delta} \begin{pmatrix} z^2\Delta + x^2 & xy \\ xy & z^2\Delta + y^2 \end{pmatrix}$$

and observe that this matrix is symmetric and has determinant $\det K[X]_\sim = z^2(z^2 + \frac{x^2+y^2}{\Delta}) \neq 0 \iff z \neq 0$. Thus as long as $z \neq 0$ the function

$$\mathcal{H}_0 = \frac{1}{2}\langle p_0, p_0 \rangle + \frac{1}{2}\langle K[X]_\sim \cdot q, q \rangle$$

could be regarded as the Hamiltonian describing a harmonic oscillator motion of the confined charged particles a, b, α, β in a magnetic field with the frequencies

$$\{\omega[X]_\sim : \det(K[X]_\sim - \omega[X]_\sim) = 0\} = \{z^2\Delta + x^2 + y^2, z^2\Delta\}$$

being allowed some internal degrees of freedom. (It would be interesting to know whether this is a physically relevant example?)

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