Applications of Sub-Supersolution Theorems to Singular Nonlinear Elliptic Problems

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Abstract
The results presented here were motivated by several recent papers on singular boundary value problems for semilinear elliptic equations with convection terms. We present extensions which cover singular nonlinear equations (mainly equations involving the \( p \)-Laplacian) containing convection terms. The results obtained are proved using sub- and supersolution theorems (motivated by the results in [18, 19, 20, 23]) and the construction of a well-ordered pair of such using a principal eigenfunction of the \( p \)-Laplacian.

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1 Introduction

Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^N \), \( N \geq 1 \). For any open set \( O \subset \mathbb{R}^m, m \in \{1, 2, \cdots\} \), a mapping \( H : \Omega \times O \to \mathbb{R} \) is said to be a Carathéodory function if, and only if:

(i) \( H(\cdot, s) \) is measurable for all \( s \in O \),

(ii) \( H(x, \cdot) \) is continuous for a.e. \( x \in \Omega \).
For $p > 1$, a constant, we consider the following problem

$$
\begin{align*}
-\Delta_p u &= g(x, u) + h(x, \nabla u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

(1.1)

where

$$
\Delta_p = \text{div}(|\nabla|^{p-2}\nabla)
$$

is the $p$-Laplace operator, and $g$ and $h$ are two Carathéodory functions defined on $\Omega \times (0, \infty)$ and $\Omega \times \mathbb{R}^N$, respectively. Moreover, $g$ is allowed to be singular in the following sense:

$$
\lim_{s \to 0} g(x, s) = \infty \quad \text{uniformly for } x \in \Omega.
$$

We cite the papers of Fulks and Maybe [11], Callegari and Nachman [4, 5] and some of their references, for giving physical situations from which problems like (1.1) arise.

The difficulties in the study of problems of the form (1.1) are the singular behavior of $g$ and the gradient dependence of the term $h$ (the convection term). In the presence of a convection term, variational methods often fail, because of the lack of a corresponding energy functional. However, employing the first (principal) eigenvalue of $-\Delta_p$ and an associated eigenfunction, we are able to construct a well-ordered pair of sub- and supersolutions to (1.1) (see Section 5). We therefore establish sub-supersolution theorems for such problems by extending some sub-supersolution theorems in [18, 19, 20] and one of the major results in [23], respectively valid for nonsingular and singular elliptic problems without convection terms. Proving these results requires several techniques. Two of the main tools are the use of the Leray-Schauder degree defined on the class $(S_+)$ (see [3]) and approximating the singular problem by nonsingular ones, obtained by considering these problems on a sequence of subdomains of $\Omega$. This suggests to consider the class of operators satisfying the $(S_+)$ condition on all subdomains of $\Omega$ motivating the definition of the class $(S_+)$ (see Definition 2.1) to which the $p$-Laplace operator belongs. The use of appropriate test functions (similar to those used in [28]) is also important to prove these sub-supersolution theorems. Finally, $W^{1,p}_0$ a priori bounds on possible solutions play an important role in our proofs because of the presence of convection terms. Such bounds will follow from certain growth conditions imposed on the nonlinear terms. Since such a priori bound results are interesting in their own right, we give such a criterion in Section 8.

Throughout the paper, we shall employ

$$
\|u\| = \left(\int_{\Omega} |\nabla u|^p \, dx\right)^{\frac{1}{p}}
$$

as the norm of $u$ in the Sobolev space $W^{1,p}_0(\Omega)$.

## 2 Sub-supersolution theorems for nonsingular problems

The problems considered in this section are slightly more general than (1.1).

We begin with assumptions on the principal operator and a discussion of the more general setting. Assume that

$$
A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N
$$
is a Carathéodory function and satisfies:

$$|A(x, ξ)| ≤ a_1(x) + b_1|ξ|^{p-1}, \quad \text{for a.e. } x ∈ Ω, \text{ all } ξ ∈ \mathbb{R}^N,$$

with $p ∈ (1, ∞)$ (fixed), $a_1 ∈ L^{\frac{p}{p-1}}(Ω)$ and $b_1 > 0$. Moreover, $A$ is assumed to be strictly monotone; i.e.,

$$(A(x, ξ) - A(x, ξ')) · (ξ - ξ') > 0,$$

for a.e. $x ∈ Ω$, all $ξ, ξ' ∈ \mathbb{R}^N$, $ξ ≠ ξ'$ and $A$ is coercive in the following sense: there exist $a_2 ∈ L^{1}(Ω)$ and $b_2 > 0$ such that

$$A(x, ξ) · ξ ≥ b_2|ξ|^p - a_2(x), \quad \text{for a.e. } x ∈ Ω, \text{ all } ξ ∈ \mathbb{R}^N.$$

Assume furthermore that the map $A : W^{1,p}_0(Ω) → (W^{1,p}_0(Ω))^*$, defined as

$$⟨Au, v⟩ := \int_Ω A(x, ∇u) · ∇v dx \quad ∀u, v ∈ W^{1,p}_0(Ω),$$

is of class $(S_+)$, where $(S_+)$ is defined as follows.

**Definition 2.1** Let $L : Ω × \mathbb{R}^N → \mathbb{R}^N$ be such that the map $L$ defined by

$$⟨Lu, v⟩ := \int_Ω L(x, u) · v dx$$

satisfies

$$L : W^{1,p}_0(Ω) → (W^{1,p}_0(Ω))^*.$$

We say that $L$ belongs to class $(S_+)$ if, and only if: for all sequences $\{u_n\}_{n∈\mathbb{N}}$ converging weakly to $u$ in $W^{1,p}_0(Ω)$, whenever $Ω' ⊂ Ω$ is such that

$$\limsup_{n→∞} \int_{Ω'} L(x, ∇u_n) · ∇(u_n - u) dx ≤ 0,$$

then

$$∇u_n → ∇u \quad \text{in } (L^p(Ω'))^N.$$

For any sequence $\{u_n\}_{n∈\mathbb{N}}$ converging weakly to a function $u$ in $W^{1,p}_0(Ω)$, it also converges to $u$ in $L^p(Ω)$, and hence in $L^p(Ω')$ for all $Ω' ⊂ Ω$. Thus,

$$∇u_n → ∇u \quad \text{in } (L^p(Ω'))^N.$$

is equivalent to

$$u_n → u \quad \text{in } W^{1,p}(Ω).$$

This helps us understand the first part of the following remark.

**Remark 2.1** The class $(S_+)$ is contained in the classical class $(S_+)$, defined in [3]. Furthermore it is the case that the $p-$Laplacian belongs to the class $(S_+)$. 


The second part of Remark 2.1 can be deduced by [8], which showed that the $p$-Laplacian
\[-\Delta_p : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*\]
belongs to the class $(S_+)$ for all domains $\Omega'$ of $\mathbb{R}^N$.

We immediately deduce that $\mathcal{A}$ is continuous, as it can be written as the composition of the continuous maps, described as follows:
\[
\mathcal{A} : W_0^{1,p}(\Omega) \to (L^p(\Omega))^N \to (L^{p'}(\Omega))^N \to \left(W_0^{1,p}(\Omega)\right)^* \ni u \mapsto \nabla u \mapsto N_A(\nabla u) \mapsto N_A(\nabla u).
\]

Here, the Nemytskii operator $N_A$, defined as
\[
N_A(w(x)) := A(x, w(x)), \quad \forall w \in (L^p(\Omega))^N,
\]
is continuous because of condition (2.1). Note that the operator $N_A(\nabla u)$ is in $(W_0^{1,p}(\Omega))^*$ in the following sense
\[
\langle N_A(\nabla u), v \rangle = \int_{\Omega} N_A(\nabla u) \cdot \nabla v dx, \quad \forall v \in W_0^{1,p}(\Omega).
\]

Let $f$ be a Carathéodory function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$. The main purpose of this section is to establish a sub-supersolution theorem for the following boundary value problem
\[
\begin{aligned}
-\text{div}(A(x, \nabla u)) &= f(x, u, \nabla u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\] 

(2.4)

We first define the concepts of subsolution and supersolution. Since $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, one can see that these concepts are identical to those in [18, 19, 20] in the case that $f$ is independent of the third variable and $f(\cdot, u) \in \left(W_0^{1,p}(\Omega)\right)^*$, where $u$ is either a sub- or a supersolution as defined below.

**Definition 2.2** The function $u \in W^{1,p}(\Omega)$ is a supersolution of (2.4) if, and only if:

(i) $u_{|\partial \Omega} \geq 0$,

(ii) $\forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), v \geq 0$,

\[
\int_{\Omega} A(x, \nabla u) \cdot \nabla v dx \geq \int_{\Omega} f(x, u, \nabla u)v dx.
\]

**Definition 2.3** The function $u \in W^{1,p}(\Omega)$ is a subsolution of (2.4) if, and only if:

(i) $u_{|\partial \Omega} \leq 0$,

(ii) $\forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), v \geq 0$,

\[
\int_{\Omega} A(x, \nabla u) \cdot \nabla v dx \leq \int_{\Omega} f(x, u, \nabla u)v dx.
\]
Remark 2.2 In the above definitions the case that
\[ \int_\Omega f(x, u, \nabla u)vdx = \pm \infty, \]
for some \( v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \), is permissible. However, we shall impose a growth condition on \( f \) so that the integral above is finite for all \( v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \).

A solution of (2.4) is defined as follows.

Definition 2.4 The function \( u \in W^{1,p}_0(\Omega) \) is a solution of (2.4) if, and only if:
\[ \int_\Omega A(x, \nabla u) \cdot \nabla vdx = \int_\Omega f(x, u, \nabla u)vdx \]
for all \( v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \).

Assume that (2.4) has \( k \) subsolutions \( u_1, \cdots, u_k, k \geq 1 \), and \( l \) supersolutions \( \overline{u}_1, \cdots, \overline{u}_l, l \geq 1 \), all of which belong to \( C^1(\overline{\Omega}) \), such that
\[ u := \max\{u_1, \cdots, u_k\} \leq \underline{u} := \min\{\overline{u}_1, \cdots, \overline{u}_l\} \quad \text{in } \Omega, \]
and there exist a function \( a_3 \in L^{p-1}(\Omega) \) and a constant \( b_3 > 0 \) such that
\[ |f(x, s, \xi)| \leq a_3(x) + b_3|\xi|^p, \quad (2.5) \]
for all \( s \in [\underline{u}_0(x), \overline{u}_0(x)], x \in \Omega \), where
\[ \underline{u}_0 := \min\{u_1, \cdots, u_k\}, \quad \overline{u}_0 := \max\{\overline{u}_1, \cdots, \overline{u}_l\}. \]

For each \( (x, s) \in \Omega \times \mathbb{R} \), define
\[ \gamma_i(x, s) := \begin{cases} \overline{u}(x), & s > \overline{u}(x), \\ \underline{u}(x), & s \leq \overline{u}(x) \leq u_i(x), \\ u_i(x), & s < \underline{u}(x), \end{cases} \]
for \( i = 1, \cdots, k \)
\[ \gamma_j(x, s) := \begin{cases} \overline{u}(x), & s > \overline{u}(x), \\ \underline{u}(x), & s \leq \overline{u}(x) \leq u_j(x), \\ u_j(x), & s < \underline{u}(x), \end{cases} \]
and for all \( j = 1, \cdots, l \)
\[ \overline{\gamma}_j(x, s) := \begin{cases} \overline{u}_j(x), & s > \overline{u}_j(x), \\ \underline{u}(x), & s \leq \overline{u}_j(x) \leq \overline{u}(x), \\ u(x), & s < \underline{u}(x). \end{cases} \]


Then for all \( u \in W^{1,p}(\Omega) \), the functions
\[
\begin{align*}
x & \mapsto \gamma(x, u(x)), \\
x & \mapsto \gamma_i(x, u(x)), \quad i = 1, \cdots, k, \\
x & \mapsto \overline{\gamma}_j(x, u(x)), \quad j = 1, \cdots, l
\end{align*}
\]
belong to \( W^{1,p}(\Omega) \). For each \( n \in \mathbb{N} \), let
\[
h_n : \mathbb{R}^N \to \mathbb{R}^N
\]
be defined as
\[
h_n(\xi) := \begin{cases} 
\xi, & |\xi| \leq n, \\
\frac{n}{|\xi|} \xi, & |\xi| > n,
\end{cases}
\]
for all \( \xi \in \mathbb{R}^N \).

We now consider the auxiliary problem
\[
\begin{cases}
-\text{div}(A(x, \nabla u)) = \tilde{f}_n(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(2.6)

where
\[
\tilde{f}_n(x, u) := f(x, \gamma(x, u), h_n(\nabla \gamma(x, u)))
+ \sum_{i=1}^k |f(x, \gamma_i(x, u), h_n(\nabla \gamma_i(x, u))) - f(x, \gamma_i(x, u), h_n(\nabla \gamma_i(x, u)))|
- \sum_{j=1}^l |f(x, \gamma_j(x, u), h_n(\nabla \gamma_j(x, u))) - f(x, \overline{\gamma}_j(x, u), h_n(\nabla \overline{\gamma}_j(x, u)))|.
\]

For each \( n \geq 1 \), define
\[
\mathcal{F}_n : W^{1,p}_0(\Omega) \to \left(W^{1,p}_0(\Omega)\right)^*
\]
as
\[
\langle \mathcal{F}_n(u), v \rangle := \int_\Omega f_n(x, u)v dx \quad \forall u, v \in W^{1,p}_0(\Omega).
\]

**Lemma 2.1** \( \mathcal{F}_n \) is demicontinuous for all \( n \geq 1 \). That is, if \( u_m \to u \) in \( W^{1,p}_0(\Omega) \), then
\[
\lim_{m \to \infty} \langle \mathcal{F}_n(u_m), v \rangle = \langle \mathcal{F}_n(u), v \rangle,
\]
for each \( v \in W^{1,p}_0(\Omega) \).

**Proof.** The lemma is proved by using (2.5), the boundedness of \( h_n, n \geq 1 \), the continuity of \( \gamma, \gamma_i \) and \( \overline{\gamma}_j \) from \( W^{1,p}(\Omega) \) to \( W^{1,p}(\Omega) \), \( 1 \leq i \leq k, 1 \leq j \leq l \), and Lebesgue’s dominated convergence theorem.

**Lemma 2.2** For all \( n \geq 1 \), \( \mathcal{A} - \mathcal{F}_n \) is of class \( (S_+) \).
Proof. Assuming that
\[ u_m \rightharpoonup u \quad \text{in} \quad W^{1,p}_0(\Omega), \]
\[ u_m \rightarrow u \quad \text{in} \quad L^p(\Omega), \]
we may use (2.5), noting that all functions \( \gamma(x,s), \gamma_i(x,s) \) and \( \gamma_j(x,s), 1 \leq i \leq k, 1 \leq j \leq l \) have their range in the interval \([u_0(x), u_0(x)]\) for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R} \), the boundedness of \( h_n \) and Hölder’s inequality to show that
\[ \lim_{m \to \infty} |\langle F_n(u_m), u_m - u \rangle| = 0. \]
Hence, the fact that
\[ \limsup_{m \to \infty} \langle A(u_m) - F_n(u_m), u_m - u \rangle \leq 0 \]
implies that
\[ \limsup_{m \to \infty} \langle A(u_m), u_m - u \rangle \leq 0, \]
and hence \( u_m \rightarrow u \) in \( W^{1,p}_0(\Omega) \) because \( A \) is of class \((S_+), \) which is contained is the class \((S_{++}). \)

Fix \( n \geq 1. \) Let \( \mathcal{G} \) be the family of demicontinuous operators of class \((S_+), \)
\[ G : B_{R_n} \to \left( W^{1,p}_0(\Omega) \right)^*, \]
where \( R_n \) is a large positive number and
\[ B_{R_n} = \{ u \in W^{1,p}_0(\Omega) : ||u|| \leq R_n \}. \]
Let \( \mathcal{H} \) be the class of affine homotopies in \( \mathcal{G} \) and let \( J \) be the dual mapping from \( W^{1,p}_0(\Omega) \) to \( \left( W^{1,p}_0(\Omega) \right)^*. \) According to Theorem 4 in [3], there exists one and only one degree function \( \deg \) on \( \mathcal{G} \) which is normalized by the map \( J \) and invariant under \( \mathcal{H}. \)

Using (2.3), we see that
\[ H_1 : [0,1] \times B_{R_n} \rightarrow \left( W^{1,p}_0(\Omega) \right)^*, \]
\[ H_1(t,u) = (1-t)A(u) + tJ(u) \]
is an affine homotopy when \( R_n \) is large enough. This yields
\[ \deg(A, B_{R_n}, 0) = 1. \]
On the other hand, the map
\[ H_2 : [0,1] \times B_{R_n} \rightarrow \left( W^{1,p}_0(\Omega) \right)^*, \]
\[ H_2(t,u) = (1-t)A(u) + tF_n(u) \]
is also in \( \mathcal{H}. \) In fact, if we assume there are \( t \in [0,1] \) and \( u \in \partial B_{R_n} \) such that
\[ H_2(t,u) = 0, \]
or equivalently,

\[ \int_{\Omega} A(x, \nabla u) \cdot \nabla v \, dx - t \int_{\Omega} \tilde{f}_n(x, u) v \, dx = 0, \]  

(2.7)

for all \( v \in W_0^{1, p}(\Omega) \), then, using \( v = u \) as the test function in (2.7), together with the fact that \( |h_n| \) is bounded by \( n \), we have

\[ \int_{\Omega} b_2|\nabla u|^p \, dx \leq \int_{\Omega} \left[ a_2 + ((1 + 2k + 2l)a_3 + 2(1 + 2k + 2l)|a_3|)|u| \right] \, dx, \]

which is impossible because \( R_n \) was chosen large. By Theorem 4 in [3],

\[ \text{deg}(H_2(t, \cdot), B_{R_n}, 0) \]

is well-defined and invariant as \( t \) varies in \([0, 1]\). This gives

\[ \text{deg}(\mathcal{A} - \mathcal{F}_n, B_{R_n}, 0) = \text{deg}(\mathcal{A}, B_{R_n}, 0) = 1, \]

and therefore the equation \( \mathcal{A} - \mathcal{F}_n = 0 \) has a solution \( u_n \in W_0^{1, p}(\Omega) \).

**Remark 2.3** It is not surprising that \( u_n \), obtained in the previous paragraph, is essentially bounded, although \( f(\cdot, u_n, \nabla u_n) \) is bounded in terms of \( |\nabla u_n|^p \) (see condition (2.5)). In earlier work of Ladyzhenskaya and Ural’tseva [15] such terms are not included. The main reason yielding \( L^\infty \) bounds is that we have replaced \( f \) by \( f_n \), which satisfies the Leray-Lions conditions (see [15]), because of the boundedness of \( h_n \).

**Lemma 2.3** For each \( n \geq 1 \), \( u_n \) is essentially bounded.

**Proof.** Since all functions \( \gamma(x, w), \gamma(x, w), \gamma(x, w), 1 \leq i \leq k, 1 \leq j \leq l, \) are uniformly bounded with respect to \( x \in \Omega, w \in W^{1, p}(\Omega) \) and since \( h_n(\xi) \) is uniformly bounded with respect to \( \xi \in \mathbb{R}^N, \) \( |\tilde{f}_n(x, w)| \) is dominated by

\[ (1 + 2k + 2l)a_3(x) + M_n \]

for a.e. \( x \in \Omega, \) all \( w \in W^{1, p}(\Omega) \). It follows that \( u_n \) satisfies

\[ |-\text{div}A(x, \nabla u_n)| \leq (1 + 2k + 2l)a_3(x) + M_n, \]

where \( k \) and \( l \) are, respectively, the number of sub- and supersolutions.

Using the monotonicity condition satisfied by \( A \) (condition (2.2)) and the weak comparison principle, we conclude that

\[ \mu_n(x) \leq u_n(x) \leq \rho_n(x) \quad \text{for a.e. } x \in \Omega, \]

where \( \rho_n \) is the solution of

\[ \begin{align*}
  -\text{div}A(x, \nabla \rho_n) &= (1 + 2k + 2l)a_3(x) + M_n \quad \text{in } \Omega, \\
  \rho_n &= 0 \quad \text{on } \partial \Omega,
\end{align*} \]

and \( \mu_n \) is the solution of

\[ \begin{align*}
  -\text{div}A(x, \nabla \mu_n) &= -(1 + 2k + 2l)a_3(x) - M_n \quad \text{in } \Omega, \\
  \mu_n &= 0 \quad \text{on } \partial \Omega.
\end{align*} \]
Since $\rho_n$ and $\mu_n$ are bounded (see [15]), so is $u_n$.

We make the convention that the assertion $w \in [w_1, w_2]$, where $w, w_1$ and $w_2$ are measurable and defined on $\Omega$, is to be understood to mean

$$w_1(x) \leq w(x) \leq w_2(x),$$

for a.e. $x \in \Omega$.

Lemma 2.4 For all $n \geq \max\{\|\nabla u_i\|_{L^\infty(\Omega)}, \|\nabla u_j\|_{L^\infty(\Omega)} : 1 \leq i \leq k, 1 \leq j \leq l\}, u_n \in [u_0, \overline{u}]$.

Proof. Fix $i \in \{1, \cdots, k\}$. Since $(u_n - \underline{u})^- \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ (see Lemma 2.3 for the boundedness of $(u_n - \underline{u})$), it is an admissible test function for (2.6). We have

$$\int_\Omega A(x, \nabla u_n) \cdot \nabla (u_n - \underline{u})^- \ dx = \int_\Omega \tilde{f}_n(x, u_n) (u_n - \underline{u})^- \ dx \geq \int_\Omega [f(x, u, \nabla u) + |f(x, u, \nabla u) - f(x, u_i, \nabla u_i)|] (u_n - \underline{u})^- \ dx \geq \int_\Omega A(x, \nabla u_i) \nabla (u_n - \underline{u})^- \ dx,$$

where the first inequality follows from the fact that

$$n \geq \max\{\|\nabla u_i\|_{L^\infty(\Omega)}, \|\nabla u_j\|_{L^\infty(\Omega)} : 1 \leq i \leq k, 1 \leq j \leq l\}. \tag{2.8}$$

Therefore,

$$\int_\Omega [A(x, \nabla u_n) - A(x, \nabla u_i)] \cdot \nabla (u_n - \underline{u})^- \ dx \geq 0,$$

which yields $(u_n - \underline{u})^- = 0$ and, hence, $u_n \geq u_i$, by using the formula

$$\nabla (u_n - \underline{u})^- = \begin{cases} -\nabla (u_n - \underline{u})^- & u_n - \underline{u} \leq 0, \\ 0 & u_n - \underline{u} \geq 0, \end{cases}$$

and the strict monotonicity of $A$.

Similarly, using the test function $(u_n - \overline{u})^+$ in (2.6) gives $u \leq \overline{u}$ for any $j \in \{1, \cdots, l\}$.

Without loss of generality, we may assume that Lemma 2.4 is valid for all $n \geq 1$. Thus, $u_n$ solves

$$\begin{cases} -\text{div}(A(x, \nabla u_n)) = f(x, u_n, h_n(\nabla u_n)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \tag{2.8}$$

for all $n \geq 1$.

Lemma 2.5 $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $W^{1,p}_0(\Omega)$. 

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Proof. By Lemma 2.4, the sequence \( \{\|u_n\|_{L^\infty(\Omega)}\}_{n \in \mathbb{N}} \) is uniformly bounded by the maximum of \( \|\tilde{u}\|_{L^\infty(\Omega)} \) and \( \|\tilde{u}\|_{L^\infty(\Omega)} \). Thus, the function

\[
v_t = e^{\tilde{u}_t} u_n,
\]

whose \( i \)-th partial derivative is

\[
\frac{\partial v_t}{\partial x_i} = e^{\tilde{u}_t} (1 + 2tu_n^2) \frac{\partial u_n}{\partial x_i}, \quad i \in \{1, 2, \cdots, N\},
\]

belongs to \( W^{1,p}_0(\Omega) \), for any positive real number \( t \). Using \( v_t \) as the test function in (2.8), we have

\[
\int_{\Omega} A(x, \nabla u_n) \cdot \nabla v_t \, dx = \int_{\Omega} f(x, u_n, h_n(\nabla u_n)) v_t \, dx,
\]

which is equivalent to

\[
\int_{\Omega} e^{\tilde{u}_t} (1 + 2tu_n^2) A(x, \nabla u_n) \cdot \nabla u_n \, dx = \int_{\Omega} e^{\tilde{u}_t} f(x, u_n, h_n(\nabla u_n)) u_n \, dx.
\]

This, together with conditions (2.3) and (2.5), shows that

\[
\int_{\Omega} e^{\tilde{u}_t} (1 + 2tu_n^2) |\nabla u_n|^p \, dx \leq \int_{\Omega} e^{\tilde{u}_t} (a_3 + b_3|\nabla u_n|^p) |u_n|^\epsilon \, dx.
\]

Again, since, \( \|u_n\|_{L^\infty(\Omega)} \) is uniformly bounded, we can find a constant \( C \) such that

\[
\int_{\Omega} b_2 e^{\tilde{u}_t} (1 + 2tu_n^2) |\nabla u_n|^p \, dx \leq C + b_3 \int_{\Omega} e^{\tilde{u}_t} |\nabla u_n|^p |u_n| \, dx
\]

\[
\leq C + b_3 \int_{\Omega} e^{\tilde{u}_t} |\nabla u_n|^p \left( \frac{\epsilon}{4} + \frac{u_n^2}{\epsilon} \right) \, dx,
\]

where \( \epsilon \) is an arbitrary positive number. We now choose \( \epsilon \) so small that

\[
\frac{b_3 \epsilon}{4} < \frac{b_2}{2}
\]

and then choose \( t \) so that

\[
2tb_2 = \frac{b_3}{\epsilon}
\]

and obtain

\[
\int_{\Omega} \frac{b_2}{2} e^{\tilde{u}_t} |\nabla u_n|^p \, dx \leq C.
\]

It follows that \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( W^{1,p}_0(\Omega) \).

Since \( W^{1,p}_0(\Omega) \) is reflexive, we can find a function \( u \in W^{1,p}_0(\Omega) \) and a subsequence of \( \{u_n\}_{n \in \mathbb{N}} \), still denoted by \( \{u_n\}_{n \in \mathbb{N}} \), such that

\[
u_n \rightharpoonup u \quad \text{in} \ W^{1,p}_0(\Omega), \\
u_n \rightarrow u \quad \text{in} \ L^p(\Omega), \\
u_n \rightarrow u \quad \text{a.e. in} \ \Omega.
\]

Here, we have used the compact embedding from \( W^{1,p}_0(\Omega) \) to \( L^p(\Omega) \) and Theorem 1.Q in [28].
Lemma 2.6 The sequence \( \{u_n\}_{n \in \mathbb{N}} \) converges to \( u \) in \( W^{1,p}_0(\Omega) \).

Proof. For \( n \geq 1 \), using
\[
e^{(u_n-u)^2}(u_n-u) \in W^{1,p}_0(\Omega)
\]
as a test function in (2.8) and using condition (2.3), we have
\[
\int_{\Omega} (1 + 2t(u_n - u)^2)e^{(u_n-u)^2} A(x, \nabla u_n) \cdot \nabla (u_n - u) dx
\]
\[
= \int_{\Omega} e^{(u_n-u)^2} f(x, u_n, h_n(\nabla u_n))(u_n - u) dx
\]
\[
\leq \int_{\Omega} e^{(u_n-u)^2}(a_3 + b_3|\nabla u_n|^p)|u_n - u|dx
\]
\[
= \int_{\Omega} e^{(u_n-u)^2}a_3|u_n - u|dx + \int_{\Omega} b_3e^{(u_n-u)^2} |\nabla u_n|^p|u_n - u|dx.
\]
Using condition (2.3), we see that
\[
\int_{\Omega} b_3e^{(u_n-u)^2} |\nabla u_n|^p|u_n - u|dx
\]
\[
\leq \int_{\Omega} b_3e^{(u_n-u)^2} |\nabla u_n|^p \left( \frac{\varepsilon}{2} + \frac{1}{2e}(u_n - u)^2 \right) dx
\]
\[
\leq \int_{\Omega} b_3 \left( a_2 + A(x, \nabla u_n) \cdot \nabla u_n \right) \left( \frac{\varepsilon}{2} + \frac{1}{2e}(u_n - u)^2 \right) dx,
\]
and consequently,
\[
\int_{\Omega} (1 + 2t(u_n - u)^2)e^{(u_n-u)^2} A(x, \nabla u_n) \cdot \nabla (u_n - u) dx
\]
\[
\leq \int_{\Omega} e^{(u_n-u)^2}a_3|u_n - u|dx
\]
\[
+ \frac{eb_3}{2b_2} \int_{\Omega} e^{(u_n-u)^2}a_2|u_n - u|dx + \frac{eb_3}{2eb_2} \int_{\Omega} e^{(u_n-u)^2}a_2(u_n - u)^2 dx
\]
\[
+ \frac{eb_3}{2b_2} \int_{\Omega} e^{(u_n-u)^2} A(x, \nabla u_n) \cdot \nabla u_n dx
\]
\[
+ \frac{eb_3}{2b_2} \int_{\Omega} e^{(u_n-u)^2} A(x, \nabla u_n) \cdot \nabla udx
\]
\[
+ \frac{b_3}{2eb_2} \int_{\Omega} e^{(u_n-u)^2}(u_n - u)^2A(x, \nabla u_n) \cdot \nabla (u_n - u) dx
\]
\[
+ \frac{b_3}{2eb_2} \int_{\Omega} e^{(u_n-u)^2}(u_n - u)^2A(x, \nabla u_n) \cdot \nabla udx.
\]
We next choose
\[
t = \frac{b_3}{4eb_2}
\]
and then combine the left hand side of the above inequality with the fourth and sixth summands of the right hand side to obtain

\[
\left(1 - \frac{b_1}{2b_2}\right) \int_{\Omega} e^{\epsilon(a_n-u)^2} A(x, \nabla u_n) \cdot \nabla (u_n - u) \, dx \\
\leq \int_{\Omega} e^{\epsilon(a_n-u)^2} a_3 |u_n - u| \, dx \\
+ \frac{b_1}{2b_2} \int_{\Omega} e^{\epsilon(a_n-u)^2} a_2 \, dx + \frac{b_3}{2b_2} \int_{\Omega} e^{\epsilon(a_n-u)^2} a_2 (u_n - u)^2 \, dx \\
+ \frac{b_3}{2b_2} \int_{\Omega} e^{\epsilon(a_n-u)^2} A(x, \nabla u_n) \cdot \nabla u \, dx \\
+ \frac{b_3}{2b_2} \int_{\Omega} e^{\epsilon(a_n-u)^2} (u_n - u)^2 A(x, \nabla u_n) \cdot \nabla u \, dx.
\]

Using Hölder’s inequality, Lebesgue’s dominated convergence theorem, the boundedness of \{u_n\}_{n \in \mathbb{N}} in \(W^{1,p}_0(\Omega)\), and condition (2.1), and letting \(n\) tend to \(\infty\), we have

\[
\limsup_{n \to \infty} \left(1 - \frac{b_1}{2b_2}\right) \int_{\Omega} e^{\epsilon(a_n-u)^2} A(x, \nabla u_n) \cdot \nabla (u_n - u) \, dx \leq \epsilon \mathcal{C},
\]

(2.9)

where \(\mathcal{C}\) is a positive number depending on \(a_i, b_i, i = 1, 2, 3\), and \(\sup\{\|u_n\| : n \in \mathbb{N}\}\).

On the other hand, the quantity

\[
\left| \int_{\Omega} (e^{\epsilon(a_n-u)^2} - 1) A(x, \nabla u) \cdot \nabla (u_n - u) \, dx \right|
\]

is bounded from above by

\[
\left( \int_{\Omega} (e^{\epsilon(a_n-u)^2} - 1) A(x, \nabla u) \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} (\nabla (u_n - u))^p \, dx \right)^{\frac{1}{p}},
\]

which tends to 0 as \(n \to \infty\) because of the dominated convergence theorem and the boundedness of \(\{u_n - u\}_{n \in \mathbb{N}}\) in both spaces \(L^p(\Omega)\) and \(W^{1,p}_0(\Omega)\). Thus, the quantity in (2.10) converges to 0 and, therefore,

\[
\lim_{n \to \infty} \int_{\Omega} e^{\epsilon(a_n-u)^2} A(x, \nabla u_n) \cdot \nabla (u_n - u) \, dx \\
= \lim_{n \to \infty} \int_{\Omega} A(x, \nabla u) \cdot \nabla (u_n - u) \, dx = 0.
\]

The last equality holds because \(u_n \to u\) in \(W^{1,p}_0(\Omega)\). This, together with (2.9), implies that

\[
\limsup_{n \to \infty} \left(1 - \frac{b_1}{2b_2}\right) \int_{\Omega} e^{\epsilon(a_n-u)^2} [A(x, \nabla u_n) - A(x, \nabla u)] \cdot \nabla (u_n - u) \, dx \\
\leq \epsilon \mathcal{C}.
\]
Since \( e^{(u_n - u)^2} > 1 \) for all \( t > 0, n \geq 1 \) and a.e. \( x \in \Omega \),

\[
\limsup_{n \to \infty} \left( 1 - \frac{e^{b_3}}{2b_3} \right) \int_\Omega [A(x, \nabla u_n) - A(x, \nabla u)] \cdot \nabla (u_n - u) \, dx \leq \epsilon C.
\]

Letting \( \epsilon \to 0^+ \), we obtain

\[
\limsup_{n \to \infty} \int_\Omega A(x, \nabla u_n) \cdot \nabla (u_n - u) \, dx \leq 0,
\]

and hence \( u_n \to u \) in \( W^{1,p}_0(\Omega) \) because \( \mathcal{A} \) is of class \((S_+)\).

Applying Theorem 1.1 in [28], we can find a function \( w \in L^p(\Omega) \) and a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \) such that \( \nabla u_n \to \nabla u \) and

\[
|\nabla u_n(x)| \leq w(x),
\]

for a.e. \( x \in \Omega \). Fix \( v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \). Using the inequality above and Lebesgue’s dominated convergence theorem and letting \( n \to \infty \) in the equation

\[
\int_\Omega A(x, \nabla u_n) \cdot \nabla v \, dx = \int_\Omega f(x, u_n, h_n(\nabla u_n)) v \, dx,
\]

we see that \( u \) is a weak solution of (2.4). We have proved the following theorem.

**Theorem 2.1** Assume that problem (2.4) has \( k \) subsolutions \( u_i, i = 1, \ldots, k \), and \( l \) supersolutions \( \overline{u}_i, \ldots, \overline{u}_l, k, l \geq 1 \), all of which belong to \( C^1(\Omega) \), such that

\[
\underline{u} := \max\{u_i : i = 1, \ldots, k\} \leq \overline{u} := \min\{\overline{u}_j : j = 1, \ldots, l\} \quad \text{in} \ \Omega.
\]

Assume further that there exist a function \( a_3 \in L^\frac{p}{p-1}(\Omega) \) and a constant \( b_3 > 0 \) such that

\[
|f(x, s, \xi)| \leq a_3(x) + b_3|\xi|^p,
\]

for all \( s \in [u_0(x), \overline{u}_0(x)] \), for a.e. \( x \in \Omega \), where

\[
\begin{align*}
\underline{u}_0 &:= \min\{u_1, \ldots, u_k\}, \\
\overline{u}_0 &:= \max\{\overline{u}_1, \ldots, \overline{u}_l\}.
\end{align*}
\]

Then (2.4) has a solution \( u \in W^{1,p}_0(\Omega) \) satisfying

\[
\underline{u}(x) \leq u(x) \leq \overline{u}(x) \quad \text{in} \ \Omega.
\]

**Remark 2.4** The requirement that all sub-supersolutions in Theorem 2.1 belong to \( C^1(\Omega) \) can be replaced by the condition that they are in \( W^{1,\infty}(\Omega) \).
Remark 2.5 Although the obtained solution $u$ in Theorem 2.1 is a subsolution or a supersolution of (2.4), we cannot use it as a subsolution or a supersolution when applying this theorem because $u$ is not in $C^1(\Omega)$. Since we want to consider $u$ as a subsolution and a supersolution later in the next section, we wish to employ some regularity results to study the smoothness of $u$. However, it is known from regularity theory that, under some additional conditions on $A$ so that (2.4) is uniformly elliptic, $u \in C^{1,\beta}(\Omega)$, $\beta > 0$ because $u$ is bounded by $\underline{u}$ and $\overline{u}$ (see [15, 21]). This does not imply that $u \in C^1(\Omega)$. We thus replace the condition that all given sub-supersolutions are in $C^1(\Omega)$ by the weaker one that they belong to $C^{1,\beta}(\Omega)$. That this may be accomplished follows by using approximation techniques.

From now on, assume $A$ that satisfies some additional conditions so that Lieberman’s regularity results hold; namely, $A$ is differentiable (except possible at $\xi = 0$) and there exist $b_3, b_5 > 0$ such that

$$\sum_{i,j = 1}^{N} a_{ij}(x,\xi)\eta_i\eta_j \geq b_3|\xi|^{p-2}|\eta|^2, \quad \forall \eta \in \mathbb{R}^N$$  \hspace{1cm} (2.11)

and

$$|a_{ij}(x,\xi)| \leq b_5|\xi|^{p-2} \quad \forall 1 \leq i, j \leq N,$$  \hspace{1cm} (2.12)

for $x \in \Omega, \xi \in \mathbb{R}^N \setminus \{0\}$. Here, $a_{ij}$ denotes $\frac{\partial^2 A_i(x)}{\partial \xi_j}$ and $A_i$ is the $i$th component of $A$, $1 \leq i, j \leq N$.

Theorem 2.2 Assume that problem (2.4) has $k$ subsolutions $\underline{u}_i$, $i = 1, \cdots, k$, and $l$ supersolutions $\overline{u}_1, \cdots, \overline{u}_l$, $k, l \geq 1$, all of which belong to $C^1(\Omega)$, such that

$$\underline{u} := \max\{\underline{u}_i : i = 1, \cdots, k\} \leq \overline{u} := \min\{\overline{u}_j : j = 1, \cdots, l\} \quad \text{in} \ \Omega.$$

Assume further that there exist a function $a_3 \in L^{\frac{p}{p-1}}(\Omega)$ and a constant $b_3 > 0$ such that

$$|f(x,s,\xi)| \leq a_3(x) + b_3|\xi|^p,$$

for all $s \in [\underline{u}_0(x), \overline{u}_0(x)]$, for a.e. $x \in \Omega$, where

$$\underline{u}_0 := \min\{\underline{u}_1, \cdots, \underline{u}_k\}, \quad \overline{u}_0 := \max\{\overline{u}_1, \cdots, \overline{u}_l\}.$$

Then (2.4) has a solution $u \in C^{1,\beta}(\Omega)$, for some $\beta > 0$, satisfying

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x) \quad \text{in} \ \Omega.$$

Proof. Let $\{\Omega_n\}_{n \geq 0}$ be the sequence of smooth subdomains of $\Omega$ such that

$$\overline{\Omega}_n \subset \Omega_{n+1}, \quad n \geq 1$$

and

$$\bigcup_{n \geq 0} = \Omega.$$
For any $n \geq 1$, the problem

\[
-\text{div}A(x, \nabla (v + u)) = f(x, v + u, \nabla (v + u)) \quad \text{in } \Omega, \\
\quad \text{on } \partial \Omega_n
\]  

(2.13)

has $u_i - u$ and $\bar{u}_j - u$ as subsolution and supersolution respectively, $1 \leq i \leq k$, $1 \leq j \leq l$. Although these functions do not belong to $C^1(\bar{\Omega}_n)$, we might apply Theorem 2.1 to find a weak solution $v_n$, whose range is in $[0, \bar{u} - u]$, of (2.13) because of the observation in Remark 2.4. Define $v_0 = 0$ on $\Omega \setminus \Omega_n$. Employing the test function $e^{\phi_n^2}v_n \in W_0^{1,p}(\Omega_n) \cap L^\infty(\Omega_n)$ for (2.13) and repeating the arguments in Lemma 2.5, we can show that $\{v_n\}_{n \in \mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, for all $n \in \mathbb{N}$, if $u_n$ is defined as $v_n + u$, then $u_n$ solves

\[
-\text{div}A(x, \nabla u_n) = f(x, u_n, \nabla u_n) \quad \text{in } \Omega, \\
\quad u_n = u \quad \text{in } \Omega \setminus \Omega_n,
\]

and the sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$.

Since $W_0^{1,p}(\Omega)$ is reflexive, we can assume that $\{u_n\}_{n \in \mathbb{N}}$ weakly converges to $u$ in $W_0^{1,p}(\Omega)$ and converges to $u$ a.e. in $\Omega$. We have the lemma.

**Lemma 2.7** Let $K$ be the closure of an open subset of $\Omega$. Then the sequence $\{u_n\}_{n \in \mathbb{N}}$ converges to $u$ in $W_0^{1,p}(K)$.

**Proof.** Let $\varphi$ be a nonnegative function in $C_c^\infty(\Omega)$ with $\varphi = 1$ in $K$ and $\Omega'$ be the support of $\varphi$. Without loss of generality, assume that $\Omega' \subset \Omega$. Fix $n \in \mathbb{N}$. Use

\[
w_i = e^{t(u_n - u)}\varphi^p(u_n - u) \in W_0^{1,p}(\Omega_n) \cap L^\infty(\Omega_n),
\]

whose gradient is

\[
\nabla w_i = e^{t(u_n - u)}\varphi^p(2t(u_n - u)^2 + 1)\nabla(u_n - u) + p e^{t(u_n - u)}\varphi^{p-1}\nabla \varphi,
\]

where $t > 0$ will be chosen later, as the test function for (2.14) to get

\[
\int_\Omega e^{t(u_n - u)}\varphi^p[2t(u_n - u)^2 + 1]A(x, \nabla u_n) \cdot \nabla(u_n - u)dx \\
+ p \int_\Omega e^{t(u_n - u)}(u_n - u)\varphi^{p-1}A(x, \nabla u_n) \cdot \nabla \varphi dx \\
= \int_\Omega f(x, u_n, \nabla u_n)e^{t(u_n - u)}\varphi^p(u_n - u)dx \\
\leq \int_\Omega |\alpha_1|e^{t(u_n - u)}\varphi^p|u_n - u|dx + b_3 \int_\Omega e^{t(u_n - u)}\varphi^p|u_n - u||\nabla u_n|^pr dx.
\]

Let $\epsilon$ be an arbitrary positive number and define

\[
c_{n, \epsilon} := \int_\Omega |\alpha_3|e^{t(u_n - u)}\varphi^p|u_n - u|dx,
\]
which converges to 0 as \( n \) tends to \( \infty \). Then the last quantity above is bounded from above by

\[
c_{n,\epsilon} + b_3 \int_\Omega e^{t(u_n-u)^2} \varphi^p \left( \frac{\epsilon}{2} + \frac{(u_n-u)^2}{2\epsilon} \right) |\nabla u_n|^p \, dx
\]

\[
\leq c_{n,\epsilon} + \frac{b_3}{b_2} \int_\Omega e^{t(u_n-u)^2} \varphi^p \left( \frac{\epsilon}{2} + \frac{(u_n-u)^2}{2\epsilon} \right) (A(\cdot, \nabla u_n) \cdot \nabla u_n + a_2) \, dx.
\]

Consequently,

\[
\int_\Omega e^{t(u_n-u)^2} \varphi^p [2t(u_n-u)^2 + 1] A(\cdot, \nabla u_n) \cdot \nabla (u_n-u) \, dx
\]

\[
\leq c_{n,\epsilon} + \frac{b_3}{b_2} \int_\Omega e^{t(u_n-u)^2} \varphi^p \left( \frac{\epsilon}{2} + \frac{(u_n-u)^2}{2\epsilon} \right) A(\cdot, \nabla u_n) \cdot \nabla u_n \, dx
\]

\[
+ \frac{b_3}{b_2} \int_\Omega e^{t(u_n-u)^2} \varphi^p \left( \frac{\epsilon}{2} + \frac{(u_n-u)^2}{2\epsilon} \right) A(\cdot, \nabla u_n) \cdot (u_n-u) \, dx,
\]

\[
+ \frac{b_3}{b_2} \int_\Omega e^{t(u_n-u)^2} \varphi^p \left( \frac{\epsilon}{2} + \frac{(u_n-u)^2}{2\epsilon} \right) a_2 \, dx.
\]

We now choose \( t = \frac{b_3}{4b_2} \), so that the inequalities above imply

\[
\limsup_{n \to \infty} \left( 1 - \frac{eb_1}{2b_2} \right) \int_\Omega e^{t(u_n-u)^2} \varphi^p A(\cdot, \nabla u_n) \cdot \nabla (u_n-u) \, dx \leq \epsilon C,
\]

where \( C \) is a positive constant depending only on \( \varphi, a_i, b_i, i = 1, 2, 3 \), and \( \sup \{ \|u_n\| : n \in \mathbb{N} \} \).

As in the proof of Lemma 2.6, we may use the boundedness of \( \{u_n\}_{n \in \mathbb{N}} \) in \( L^\infty(\Omega) \), Hölder’s inequality, and the dominated convergence theorem to see that

\[
\lim_{n \to \infty} \int_\Omega (e^{t(u_n-u)^2} - 1) \varphi^p A(\cdot, \nabla u) \cdot \nabla (u_n-u) \, dx = 0.
\]

Therefore,

\[
\lim_{n \to \infty} \int_\Omega e^{t(u_n-u)^2} \varphi^p A(\cdot, \nabla u) \cdot \nabla (u_n-u) \, dx
\]

\[
= \lim_{n \to \infty} \int_\Omega \varphi^p A(\cdot, \nabla u) \cdot \nabla (u_n-u) \, dx = 0.
\]

The last equality is obtained from the weak convergence of \( \{u_n\}_{n \in \mathbb{N}} \) to \( u \) in \( W^{1,p}(\Omega) \). Hence,

\[
\limsup_{n \to \infty} \left( 1 - \frac{eb_1}{2b_2} \right) \int_\Omega e^{t(u_n-u)^2} \varphi^p [A(\cdot, \nabla u_n) - A(\cdot, \nabla u)] \cdot \nabla (u_n-u) \, dx
\]

\[
\leq \epsilon C.
\]

Noting that the integrand in the inequality above is nonnegative, \( e^{t(u_n-u)^2} \geq 1 \) in \( \Omega \) and \( \varphi = 1 \) in \( K \), we have

\[
\limsup_{n \to \infty} \left( 1 - \frac{eb_1}{2b_2} \right) \int_K [A(\cdot, \nabla u_n) - A(\cdot, \nabla u)] \cdot \nabla (u_n-u) \, dx \leq \epsilon C,
\]
Letting $\epsilon \to 0^+$ in the inequality above, we conclude
$$\limsup_{n \to \infty} \int_K [A(x, \nabla u_n) - A(x, \nabla u)] \cdot \nabla (u_n - u) \, dx \leq 0.$$ 

Since $\mathcal{A}$ is of class $(S_+^r)$, $u_n \to u$ in $W^{1,p}(K)$.

This lemma helps us to see that
$$\int_\Omega A(x, \nabla u) \cdot \nabla \phi \, dx = \int_\Omega f(x, u, \nabla u) \phi \, dx$$
for all $\phi \in C_\infty(\Omega)$. This is also true for $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ because of the density of $C_\infty(\Omega)$ in $W_0^{1,p}(\Omega)$ and condition (2.5).

On the other hand, Theorem 1.7 in [21] can be used to deduced that $u \in C^1(\Omega)$ for some $\beta > \epsilon (0, 1)$.

### 3 Extremal solutions

We begin this section by defining the concepts of minimal and maximal solutions between a pair of functions.

**Definition 3.1** Let $\underline{u}$ and $\overline{u}$ be two measurable functions defined on $\Omega$ such that $\underline{u}(x) \leq \overline{u}(x)$ a.e. in $\Omega$. The function $u$ is said to be the minimal (or maximal) solution of (2.4) with respect to the pair $\underline{u}$ and $\overline{u}$ if, and only if:

(i) $u$ is a solution of (2.4),

(ii) $u \in [\underline{u}, \overline{u}]$ a.e. in $\Omega$,

(iii) if $w \in [\underline{u}, \overline{u}]$ is any other solution of (2.4) then

$$w(x) \geq (\text{or} \leq) u(x)$$

a.e. $x \in \Omega$.

Our main goal in this section is to show the existence of a minimal solution and a maximal solution to (2.4) with the assumption that (2.4) has several subsolutions and supersolutions. The proof of the result is based on the techniques in [18] with some suitable modifications.

**Theorem 3.1** Assume that problem (2.4) has $k$ subsolutions $\underline{u}_i$, $i = 1, \ldots, k$, and $l$ supersolutions $\overline{u}_1, \ldots, \overline{u}_l$, $k, l \geq 1$, all of which are in $C^1(\Omega)$, such that

$$u := \max\{\underline{u}_i : i = 1, \ldots, k\} \leq \overline{u} := \min\{\overline{u}_j : j = 1, \ldots, l\} \quad \text{in } \Omega.$$ 

Assume further that there exist a function $a_3 \in L^{\frac{N}{2+\eta}}(\Omega)$ and a constant $b_3 > 0$ such that

$$|f(x, s, \xi)| \leq a_3(x) + b_3|\xi|^p,$$
for all $s \in [u_0(x), \overline{u}_0(x)]$, for a.e. $x \in \Omega$, where

$$
u_0 := \min\{u_1, \ldots, u_k\},$$

$$\overline{u}_0 := \max\{\overline{u}_1, \ldots, \overline{u}_l\}.$$  

Then (2.4) has a minimal solution $u_*$ and a maximal solution $u^*$, both of which are in $C^1(\Omega)$, with $u(x) \leq u_*(x) \leq u^*(x) \leq \overline{u}(x)$, for a.e. $x \in \Omega$.

The proof of the existence of $u_*$ will follow from several lemmas given below. The existence of $u^*$ may be deduced in a similar fashion.

Let $U$ denote the set of $C^1(\Omega)$ solutions of (2.4) between $\underline{u}$ and $\overline{u}$; that is, $U := \{u \in C^1(\Omega) : u$ is a solution of (2.4) such that $u \in [\underline{u}, \overline{u}]\}$.

By Theorem 2.2, $U$ is nonempty.

**Lemma 3.1** $U$ is compact in $W^{1, p}_0(\Omega)$.

**Proof.** Let $M := \|\nu_0\|_{L^\infty(\Omega)} + \|\overline{u}_0\|_{L^\infty(\Omega)}$, and let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $U$. Since $u_n$ solves

$$
\begin{align*}
-\text{div}A(x, \nabla u_n) &= f(x, u_n, \nabla u_n) & \text{in } \Omega, \\
\quad u_n &= 0 & \text{on } \partial \Omega,
\end{align*}
$$

it also solves

$$
\begin{align*}
-\text{div}A(x, \nabla u_n) &= \hat{f}(x, u_n, \nabla u_n) & \text{in } \Omega, \\
\quad u_n &= 0 & \text{on } \partial \Omega,
\end{align*}
$$

for each $n \geq 1$, where

$$
\hat{f}(x, s, \xi) := \begin{cases}
  f(x, \overline{u}(x), \xi) & s \geq \overline{u}(x), \\
  f(x, s, \xi) & \underline{u}(x) \leq s \leq \overline{u}(x), \\
  f(x, u(x), \xi) & s \leq \underline{u}(x),
\end{cases}
$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$.

Applying Proposition 8.1 to the latter problem, we obtain the boundedness of $(u_n)_{n \in \mathbb{N}}$. Hence, this sequence contains a subsequence, still called $(u_n)_{n \in \mathbb{N}}$, weakly converging to some function $u$ in $W^{1, p}_0(\Omega)$. Following the same arguments as used in the proof of Lemma 2.6, we get the strong convergence of $(u_n)_{n \in \mathbb{N}}$ to $u$. Finally, the fact $u \in U$ can be obtained by employing Theorem 1.4 in [28] and the dominated convergence theorem.

**Lemma 3.2** Every chain in $U$ has a lower bound in $U$, with respect to the partial order $\leq$ (the usual partial order of real functions).
Proof. Let $S$ be a chain in $U$. Since $U$ is uniformly bounded in $L^1(\Omega)$, so is $S$. Define

$$ \delta = \inf_{u \in S} \left\{ \int_{\Omega} ud\lambda \right\} \geq \int_{\Omega} u d\lambda $$

and let $\{u_n\}_{n \in \mathbb{N}} \subset S$ be such that

$$ \int_{\Omega} u_{n-1} d\lambda \geq \int_{\Omega} u_n d\lambda \geq \delta, \quad n \geq 2, \quad (3.1) $$

$$ \int_{\Omega} u_n d\lambda \to \delta, \quad \text{as } n \to \infty. \quad (3.2) $$

Since both $u_{n-1}$ and $u_n$ are in $S$, for each $n \geq 2$, either $u_{n-1} \leq u_n$ or $u_{n-1} \geq u_n$ holds. The former case (with proper inequality) cannot occur because of (3.1). Thus,

$$ u_{n-1} \geq u_n, \quad n \geq 2. $$

Since $U$ is compact, we may assume that

$$ u_n \to u, \quad \text{in } W_0^{1,p}(\Omega) $$

and

$$ u_n \to u, \quad \text{in } L^1(\Omega), $$

for some $u \in S$. This, together with (3.2), implies

$$ \int_{\Omega} u d\lambda = \delta. $$

We now show that $u$ is a lower bound of $S$. Let $v$ be an arbitrary element of $U$. Consider two cases. If

$$ \int_{\Omega} v d\lambda = \delta, $$

then

$$ \int_{\Omega} v d\lambda \leq \int_{\Omega} u_n d\lambda, $$

and, therefore,

$$ v \leq u_n, $$

for all $n \in \mathbb{N}$ because $S$ is a chain. Thus, $v \leq u$. This and the fact that

$$ \int_{\Omega} v d\lambda = \delta = \int_{\Omega} u d\lambda $$

show $u = v$ in $\Omega$. If

$$ \int_{\Omega} v d\lambda > \delta, $$

then there exists $n^* \in \mathbb{N}$ such that

$$ \int_{\Omega} v d\lambda \geq \int_{\Omega} u_{n^*} d\lambda, $$
and hence,
\[ \nu \geq u_n \]
for all \( n \geq n^* \). This shows \( \nu \geq u \) by letting \( n \), in the inequality above, tend to \( \infty \).

Using Zorn’s Lemma, we obtain a minimal element \( u_* \) in \( U \) with respect to the partial order \( \leq \). We now show that \( u_* \) is the minimal element of \( U \). If \( u \) is an element of \( U \) such that \( u \not \geq u_* \), it and \( u_* \) may be considered as two supersolutions of (2.4). Theorem 2.2 may be applied to find \( u' \in U \) with
\[ u \leq u' \leq \min\{u, u_*\} \leq \overline{u}. \]
The minimality of \( u_* \) shows that \( u' = u_* \) and that \( u_* \) is the minimal solution of (2.4) in \( [u, \overline{u}] \).

4 Sub-supersolution theorems for singular problems

Note that Theorem 2.2 and Theorem 3.1 might not be applicable to problems of the form
\[
\begin{array}{ll}
-\Delta_p u &= f(x, u, \nabla u) \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{array}
\]  
(4.1)
when \( f(x, \cdot, \xi) \) is allowed to be singular at 0. The main reason for this is the lack of a growth condition (2.5). Since we are interested in singular problems, we establish, in this section, another version of a sub-supersolution theorem with a more relaxed growth condition on \( f \), by employing Theorem 2.2, Theorem 3.1 and the approximation technique in [23]. The result can be used to prove the existence of solutions of a class of singular problems.

**Definition 4.1** The function \( u \in W^{1,p}_{\text{loc}}(\Omega) \) is a subsolution of (4.1), in the sense of distributions, if, and only if:

(i) \( u > 0 \) in \( \Omega \\
(ii) \quad \text{for all } v \in C^\infty_0(\Omega), v \geq 0,
\[ \int_\Omega |\nabla d|^{p-2} \nabla u \cdot \nabla v dx \leq \int_\Omega f(x, u, \nabla u) v dx. \]

**Definition 4.2** The function \( u \in W^{1,p}_{\text{loc}}(\Omega) \) is a supersolution of (4.1), in the sense of distributions, if, and only if:

(i) \( u > 0 \) in \( \Omega \\
(ii) \quad \text{for all } v \in C^\infty_0(\Omega), v \geq 0,
\[ \int_\Omega |\nabla d|^{p-2} \nabla u \cdot \nabla v dx \geq \int_\Omega f(x, u, \nabla u) v dx. \]
Since both sub- and supersolutions, defined above, are in \(W^{1,p}_{\text{loc}}(\Omega)\) but might not belong to \(W^{1,p}(\Omega)\), their traces on \(\partial \Omega\) are not necessarily defined. However, this is not a problem, since we only need the traces of these functions defined on the boundaries of subdomains of \(\Omega\), instead of \(\Omega\).

**Remark 4.1** The requirement of smoothness of the test function \(v\) may be relaxed because of the density of \(C_0^\infty(\Omega')\) in \(W^{1,p}_{\text{loc}}(\Omega')\), for all open bounded and smooth domains \(\Omega'\) of \(\mathbb{R}^N\). More precisely, admissible test functions \(v \geq 0\) may be required to belong to \(W^{1,p}(\Omega) \cap L^\infty(\Omega)\), and have compact support.

The following is the main theorem in this section.

**Theorem 4.1** Let \(f\) be a Carathéodory function defined on \(\Omega \times (0, \infty) \times \mathbb{R}^N\). Assume that problem (4.1) has \(k\) subsolutions \(u_1, \ldots, u_k\) and \(l\) supersolutions \(\bar{u}_1, \ldots, \bar{u}_l\), in the sense of distributions, all belonging to \(C^1(\Omega) \cap L^\infty(\Omega)\), with

\[
\underline{u} := \max\{u_1, \ldots, u_k\} \leq \overline{u} := \min\{\bar{u}_1, \ldots, \bar{u}_l\},
\]

and that there exist a function \(a_3 \in L^p_{\text{loc}}(\Omega)\) and a constant \(b_3 > 0\) such that

\[
|f(x, s, \xi)| \leq a_3(x) + b_3|\xi|^p,
\]
for a.e. \(x \in \Omega\), all \(s \in [\underline{u}_0(x), \overline{u}_0(x)]\) where

\[
\underline{u}_0 := \min\{u_1, \ldots, u_k\}, \quad \overline{u}_0 := \max\{\bar{u}_1, \ldots, \bar{u}_l\}.
\]

Then, the first equation of (4.1) has a solution \(u \in C^1(\Omega) \cap L^\infty(\Omega)\), in the sense of distributions, with \(u \in [\underline{u}, \overline{u}]\), i.e., for all \(\phi \in C_0^\infty(\Omega)\),

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dx = \int_{\Omega} f(x, u, \nabla u) \phi dx.
\]

Moreover, if \(\underline{u}\) and \(\overline{u}\) are identically 0 on \(\partial \Omega\), then so is \(u\) and it solves (4.1).

Note that (4.2) is more general than (2.5) in the sense that \(a_3\) is allowed to be in \(L^p_{\text{loc}}(\Omega)\), rather than \(L^p(\Omega)\). This will explain (see the next section) why Theorem 4.1 is applicable to singular problems.

**Remark 4.2** Theorem 4.1 is similar to Theorem 1 in [2]. The main difference between these two theorems concerns the types of solution. Theorem 1 in [2] is concerned with classical solutions when \(p = 2\) while Theorem 4.1 provides solutions in the sense of distributions.

**Remark 4.3** The smoothness of the test function \(\phi\) in (4.3) is not important. If \(u\) is a solution of (4.1) in the sense of distributions, then (4.3) is true for all \(\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)\) with compact support.
Theorem 4.1 is, of course, also valid, if the \( p \)-Laplace operator is replaced by a more general elliptic operator satisfying the conditions used earlier.

Remark 4.4 The proof of Theorem 4.1 is based on several lemmas given below. Let \( \{ \Omega_n \}_{n \in \mathbb{N}} \) be a sequence of subdomains of \( \Omega \) such that

\[
\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n
\]

and

\[
\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots.
\]

For each \( n \in \mathbb{N}, i \in \{1, \ldots, k\} \) and \( j \in \{1, \ldots, l\}, \) \( u_i - u \) and \( \overline{u}_j - u \) are subsolutions and supersolutions, respectively, in the sense of Definition 2.2 and Definition 2.3, of

\[
\begin{cases}
-\Delta_p (v_n + u) &= f(x, v_n + u, \nabla (v_n + u)) \quad \text{in } \Omega_n, \\
v_n &= 0 \quad \text{on } \partial \Omega_n.
\end{cases}
\]  

Noting that

\[
0 = \max\{u_1 - u, \ldots, u_k - u\},
\]

and

\[
\overline{u} - u = \min\{\overline{u}_1 - u, \ldots, \overline{u}_l - u\},
\]

we may employ Theorem 3.1 to find a minimal solution \( v_n \in [0, \overline{u} - u] \) of (4.4). Define \( v_n = 0 \) on \( \Omega \setminus \Omega_n. \)

Lemma 4.1 The sequence \( \{v_n\}_{n \in \mathbb{N}} \) is increasing.

Proof. Fix \( n \geq 2. \) Since 0 is the maximum of \( u_1 - u, \ldots, u_k - u \) and \( v_n|_{\Omega_n-1} \) and \( v_{n-1} \) are two supersolutions of (4.4), we can find a solution \( w \) of (4.4) such that

\[
0 \leq w \leq \min\{v_n|_{\Omega_n-1}, v_{n-1}\} \leq \overline{u} - u,
\]

by Theorem 2.2. Since \( v_{n-1} \) is the minimal solution of (4.4) between the pair \( [0, \overline{u} - u] \),

\[
v_{n-1} \leq w \leq \min\{v_n|_{\Omega_n-1}, v_{n-1}\} \leq v_n|_{\Omega_n-1},
\]

which proves the lemma.

For all \( n \in \mathbb{N}, \) define \( u_n = v_n + u. \) Since \( \{v_n\}_{n \in \mathbb{N}} \) is increasing, so is \( \{u_n\}_{n \in \mathbb{N}}. \) Since \( u_n \in [u, \overline{u}] \) for all \( n \in \mathbb{N}, \) we can find a function \( u \in [u, \overline{u}] \) such that

\[
\lim_{n \to \infty} u_n = u
\]

for a.e. \( x \in \Omega. \) The following lemma shows \( u \in W^{1, p}_{\text{loc}}(\Omega) \) is the desired solution.

Lemma 4.2 Let \( K \subset \Omega \) be the closure of an open set. Then the sequence \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( W^{1, p}(K) \) for all such subsets.
Proof. Since \( \{u_n\}_{n \in \mathbb{N}} \) is uniformly bounded in \( L^{\infty}(\Omega) \), it is sufficient to show that \( |\nabla u_n|_{n \in \mathbb{N}} \) is bounded in \( L^p(K) \).

Let \( \varphi \geq 0 \) be in \( C_0^\infty(\Omega) \) such that \( \varphi = 1 \) in \( K \). Define, for a positive number \( t \), to be chosen later,

\[
w_t = e^{\frac{t^2}{2}} \varphi^p u_n,
\]

whose gradient is

\[
\nabla w_t = e^{\frac{t^2}{2}} \varphi^p (2tu_n^2 + 1) |\nabla u_n| dx + p e^{\frac{t^2}{2}} \varphi^{p-1} u_n |\nabla u_n| dx.
\]

Assume, without loss of generality, for all \( n \geq 1 \), that the support of \( \varphi \) is contained in \( \Omega_n \) and hence \( w_t \in W^{1,p}_0(\Omega_n) \). Now, using condition (4.2) and \( w_t \) as the test function in (4.4), we obtain

\[
\int_\Omega e^{\frac{t^2}{2}} \varphi^p (2tu_n^2 + 1) |\nabla u_n| dx + p \int_\Omega e^{\frac{t^2}{2}} \varphi^{p-1} u_n |\nabla u_n| dx
\]

\[
= \int_\Omega f(x,u_n, \nabla u_n) e^{\frac{t^2}{2}} \varphi^p u_n dx
\]

\[
\leq \int_\Omega |a_3| e^{\frac{t^2}{2}} \varphi^p |u_n| dx + b_3 \int_\Omega |\nabla u_n|^p e^{\frac{t^2}{2}} \varphi^p |u_n| dx
\]

\[
\leq \int_\Omega |a_3| e^{\frac{t^2}{2}} \varphi^p |u_n| dx + b_3 \int_\Omega |\nabla u_n|^p e^{\frac{t^2}{2}} \varphi^p \left( \frac{u_n^2}{2e} + \frac{\epsilon}{2} \right) dx,
\]

for all \( \epsilon > 0 \). Since \( \{u_n\}_{n \in \mathbb{N}} \) is uniformly bounded in \( L^{\infty}(\Omega) \), \( a_3 \in L^p_{\text{loc}}(\Omega) \) and \( \varphi \in C_0^\infty(\Omega) \), we can find two numbers \( b_{t,\varphi} \) and \( c_{t,\varphi} \) such that

\[
p e^{\frac{t^2}{2}} |u_n| \varphi \leq b_{t,\varphi},
\]

in \( \Omega \), and

\[
\int_\Omega |a_3| e^{\frac{t^2}{2}} \varphi^p |u_n| dx \leq c_{t,\varphi}.
\]

Moreover, we can choose

\[
t = \frac{b_3}{4\epsilon}
\]

so that

\[
\left( 1 - \frac{\epsilon b_3}{2} \right) \int_\Omega e^{\frac{t^2}{2}} \varphi^p |\nabla u_n|^p dx \leq c_{t,\varphi} + b_{t,\varphi} \left( \int_\Omega \varphi^p |\nabla u_n|^p dx \right)^{\frac{\epsilon b_3}{2} + 1}.
\]

We now choose \( \epsilon = \frac{1}{b_3} \) and get

\[
\frac{1}{2} \int_\Omega \varphi^p |\nabla u_n|^p dx \leq c_{t,\varphi} + b_{t,\varphi} \left( \int_\Omega \varphi^p |\nabla u_n|^p dx \right)^{\frac{1}{b_3} + 1}.
\]

Here, we have used \( e^{\frac{t^2}{2}} \geq 1 \). So, \( \int_\Omega \varphi^p |\nabla u_n|^p dx \) is bounded uniformly in \( n \). Therefore, \( |\nabla u_n|_{n \in \mathbb{N}} \) is bounded in \( L^p(K) \) because \( \varphi = 1 \) in \( K \).
We now show that \( u \in W^{1,p}_{\text{loc}}(\Omega) \). In fact, let \( \Omega' \) be such that its closure, \( \overline{\Omega'} \), is contained in \( \Omega \). Lemma 4.2 and the reflexivity of \( W^{1,p}(\Omega') \) help us to find a weakly convergent subsequence of \( \{u_n\}_{n \in \mathbb{N}} \) in \( W^{1,p}(\Omega') \). The weak limit of such a subsequence must be \( u \) because \( u_n \to u \) a.e. in \( \Omega \). Hence, \( u \in W^{1,p}(\Omega') \).

Let \( \phi \in C_0^\infty(\Omega) \) and let \( K \) denote the support of \( \phi \). Applying the arguments in the previous paragraph when \( \Omega' \) is the interior of \( K \), we have \( u_n \rightharpoonup u \) in \( W^{1,p}(\Omega') \).

The following lemma can be proved using the same arguments as Lemma 2.7.

**Lemma 4.3** \( u_n \to u \) in \( W^{1,p}(K) \).

Recall that \( K \) is the support of \( \phi \). Letting \( n \to \infty \) in the following equation
\[
\int_K |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \phi \, dx = \int_K f(x, u_n, \nabla u_n) \phi \, dx,
\]
implies that \( u \) solves (4.1) in the sense of distribution.

Fix \( x \in \Omega \) and let \( B \) be a ball containing \( x \) such that \( \overline{B} \) is contained in \( \Omega \). It follows from Theorem 1.7 in [21] and the fact \( u \in L^\infty(\Omega) \) that \( u \in C^1(B) \). This completes the proof of Theorem 4.1.

**Theorem 4.2** If all assumptions of Theorem 4.1 hold, then (4.1) has a minimal and a maximal solution with respect to the pair \( (u, \overline{u}) \).

**Proof.** Let
\[
U := \{ u \in [u, \overline{u}] : u \text{ solves (4.1) in the sense of distributions} \}
\]
be the set of solutions of (4.1) lying between \( u \) and \( \overline{u} \). Because of Theorem 4.1, \( U \) is nonempty. We may now employ the arguments in Section 3 to show the existence of a minimal solution of (4.1) between \( u \) and \( \overline{u} \).

Let \( S \) be a chain in \( U \), with respect to the partial order \( \leq \) and define \( \delta \) as
\[
\delta = \inf_{x \in S} \int_{\Omega} u \, dx \geq \int_{\Omega} u \, dx \geq -\infty.
\]
By definition of the infimum, it is possible to find a sequence \( \{u_n\}_{n \in \mathbb{N}} \subset S \) such that
\[
\int_{\Omega} u_{n+1} \, dx \leq \int_{\Omega} u_n \, dx
\]
for all \( n \geq 1 \) and
\[
\lim_{n \to \infty} \int_{\Omega} u_n \, dx = \delta.
\]
Since \( S \) is a chain, \( u_{n+1} \leq u_n \). Thus, \( \{u_n\}_{n \in \mathbb{N}} \) converges to a function \( u_* \in [u, \overline{u}] \) a.e. in \( \Omega \). Employing the arguments in Lemma 4.2 and Lemma 4.3, we may show that \( \{u_n\}_{n \in \mathbb{N}} \) is bounded and strongly converges to \( u \) in \( W^{1,p}(K) \), for any compact subset \( K \) of \( \Omega \). Hence, \( u \in U \). On the other hand, for all \( w \in S \), if
\[
\int_{\Omega} w \, dx = \delta
\]
then
\[ \int_{\Omega} w dx \leq \int_{\Omega} u_n dx, \]
for all \( n \geq 1 \). Since \( S \) is a chain, \( w \leq u_n \) for all \( n \geq 1 \). Letting \( n \to \infty \), we have \( w \leq u \). This and the fact that
\[ \int_{\Omega} w dx = \delta = \int_{\Omega} u dx \]
imply
\[ w(x) = u(x) \]
a.e. in \( \Omega \). If \( \int_{\Omega} w dx > \delta \), then it is easy to show \( w \geq u \) a.e. in \( \Omega \). Both cases show that \( u \) is a lower bound of \( S \). In other words, any chain in \( U \) has a lower bound. Using Zorn’s Lemma, we obtain a minimal element \( u^* \) in \( U \) with respect to the partial order \( \leq \).

The existence of a maximal solution to (4.1) may be established similarly.

5 Singular boundary value problems

Let \( \lambda_1 \) be the first (principal) eigenvalue of \( -\Delta_p \) and let \( \Phi \) denote an eigenfunction of \( -\Delta_p \) associated to \( \lambda_1 \); i.e., \( \Phi \) solves
\[
\begin{cases}
-\Delta_p \Phi = \lambda_1 |\Phi|^{p-2} \Phi & \text{in } \Omega, \\
\Phi = 0 & \text{on } \partial \Omega.
\end{cases}
\]
It is well-known that \( \Phi \) belongs to \( C^1(\Omega) \), that \( \Phi \) may be chosen positive in \( \Omega \) and that \( |\nabla \Phi| \) is positive on a neighborhood of \( \partial \Omega \). The existence and smoothness of \( \Phi \) follow from [21, 22]. The last fact is a corollary of Lemma A.3 in [24].

Our main purpose in this section is to employ \( \Phi \) to construct a well-ordered pair of sub-supersolutions to
\[
\begin{cases}
-\Delta_p u = g(x, u) + h(x, \nabla u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
and then seek a solution for this problem.

Here, \( g \) and \( h \) are two Carathéodory functions defined on \( \Omega \times (0, \infty) \) and \( \Omega \times \mathbb{R}^N \) respectively. We assume throughout this section that there exist \( C_1 > 0, C_2 > 0, \) and \( \alpha > 0 \) such that for a.e. \( x \in \Omega \),
\[
\begin{align*}
g(x, s) &\leq C_1 s^{-\alpha}, & \forall s > 0, \\
g(x, s) &> 1 & \forall s \in (0, 1),
\end{align*}
\]
and
\[ |h(x, \xi)| \leq C_2|\xi|^p, \quad \forall \xi \in \mathbb{R}^N. \]

Note that \( g(x, \cdot) \) is allowed to have a singularity at \( 0^+ \) for a.e. \( x \in \Omega \).

**Lemma 5.1** Problem (5.1) has a supersolution.

**Proof.** Let \( b > 1 \) be so large that
\[ tp - t - p + ta < 0, \]
where
\[ t = b^{-1} - \frac{a}{p - 1}. \]
Define
\[ \Psi_b = b\Phi^t. \]

By a straightforward calculation, we have
\[
\nabla \Psi_b = b\Phi^{t-1}\nabla \Phi,
\]
\[
-\Delta_p \Psi_b = (bt)^{p-1}\Phi^{p-t-1} [(1-t)(p-1)|\nabla \Phi|^p + \lambda_1 \Phi^p].
\]

Thus,
\[
-\Delta_p \Psi_b = (bt)^{p-1}\Phi^{p-t-1} \left[ \frac{1}{2} (1-t)(p-1)|\nabla \Phi|^p + \lambda_1 \Phi^p \right]
+ \frac{1}{2} (1-t)(p-1)(bt)^{p-t-1}\Phi^{p-t-1}|\nabla \Phi|^p
= b^\frac{1}{2} \Phi^{p-t-1} \left[ \frac{1}{2} (1-t)(p-1)|\nabla \Phi|^p + \lambda_1 \Phi^p \right] \Psi_b^{-a}
+ \frac{1}{2} (1-b^{-1} - \frac{a}{p - 1})(p-1)b \frac{a}{p - 1} \Phi^{p-t-1}|\nabla \Psi_b|^p.
\]

Since \( b \) may be chosen arbitrarily large, the last quantity above is dominated by
\[ g(x, \Psi_b) + h(x, \nabla \Psi_b). \]
In other words, \( \bar{u} = \Psi_b \) is a supersolution of (5.1) for \( b \) large.

Let \( \epsilon > 0 \) be chosen so small that
\[ \epsilon^{p-1} \lambda_1 \Phi^{p-1} + C_2|\nabla (\epsilon \Phi)|^p < 1. \]
Assuming also that
\[ \epsilon \Phi < 1, \]
we have
\[ -\Delta_p (\epsilon \Phi) - |h(x, \epsilon \Phi)| < 1 < g(x, \epsilon \Phi). \]
This gives us the following Lemma.

**Lemma 5.2** If \( 0 < \epsilon \ll 1 \), then the function \( \underline{u} = \epsilon \Phi \) is a subsolution of (5.1).
We are now in the position to prove an existence result, which is also our main theorem in this section.

**Theorem 5.1** Problem (5.1) has a minimal solution and a maximal solution with respect to the pair \((\epsilon \Phi, b \Phi')\), where \(\epsilon > 0\) is sufficiently small, \(b\) is sufficiently large, \(\Phi\) is a first (principal) positive eigenfunction of \(-\Delta_p\) and \(t = b^{-1-p^*}\).

**Proof.** It follows from Lemma 5.1 and Lemma 5.2 that

\[ u = b \Phi' \]

is a supersolution and

\[ u = \epsilon \Phi \]

is a subsolution of (5.1). Since \(t = b^{-1-p^*} < 1\), then if \(\Phi < 1\), we have \(\Phi \leq \Phi'\) and therefore, \(u \leq \Pi\). Otherwise, when \(\Phi \geq 1\), \(b \Phi' > 1 > \epsilon \Phi\). Both cases yield

\[ u \leq \Pi \]

in \(\Omega\). The theorem then follows by applying Theorem 4.2.

6 Singular problems with parameter dependent terms

Let \(g : (0, \infty) \rightarrow [0, \infty)\) be continuous and \(h : \Omega \times [0, \infty) \rightarrow \mathbb{R}\) be a Carathéodory function. Note that \(g\) might be singular at 0⁺. Motivated by [13, 23], we consider the following problem

\[
\begin{cases}
-\Delta_p u + k_1 |\nabla u|^q = k_2 g(u) + \lambda h(x, u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
\end{cases}
\]

(6.1)

where \(k_1 \geq 0\) and \(k_2 \geq 0\) are nonnegative \(L^\infty(\Omega)\) functions and \(\lambda\) is a nonnegative parameter.

**Theorem 6.1** Assume there exist \(C > 0\) and \(\alpha > 0\) such that

\[ g(s) \leq Cs^{-\alpha} \quad \forall s \in (0, \infty). \]

Then the following hold:

(i) If \(\limsup_{s \to 0^+} \frac{h(x, s)}{s^{p-1}} < \infty\) uniformly in \(x \in \Omega\), there exists \(\bar{\lambda} > 0\) such that for all \(\lambda \in [0, \bar{\lambda}]\), problem (6.1) has a solution.

(ii) If there exists \(q < p - 1\) such that

\[ 0 \leq h(x, s) \leq s^q, \quad \forall s \in [1, \infty), \]

uniformly in \(x \in \Omega\), then for all \(\lambda \geq 0\), problem (6.1) has a solution.
Proof. Employing Theorem 4.1 in [23], we can find a solution $\bar{u} \in W^{1,p}_{\text{loc}}(\Omega)$, in the sense of distributions, of
\[
\begin{aligned}
-\Delta_p \bar{u} &= k_2 g(\bar{u}) + \lambda h(x, \bar{u}) & \text{in } \Omega, \\
\bar{u} &> 0 & \text{in } \Omega, \\
\bar{u} &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]
with $\bar{u} \geq \epsilon \Phi$ for all $0 < \epsilon \ll 1$, where $\Phi$ is a principal positive eigenfunction of $-\Delta_p$. Note that the existence of $\bar{u}$ depends on the growth of $h$, described in (i) and (ii). If
\[
\limsup_{s \to 0^+} \frac{h(x, s)}{s^{p-1}} < \infty,
\]
uniformly in $x \in \Omega$, then there exists $\lambda$ such that the existence of $\bar{u}$ is guaranteed when $\lambda \in [0, \lambda]$. If there exists $q < p - 1$ such that
\[
0 \leq h(x, s) \leq s^q, \quad \forall s \in [1, \infty),
\]
uniformly in $x \in \Omega$, then for all $\lambda \geq 0$, (6.2) is solvable in the sense of distributions. Applying the regularity results as in the last part of the proof of Theorem 4.1, we have $\bar{u} \in C^1(\Omega)$. Obviously, $\bar{u}$ is a supersolution of (6.1). On the other hand, $u = \epsilon \Phi$ may be shown to be a subsolution of (6.1), as was done in Lemma 5.2. Thus, Theorem 4.1 may be used to prove the existence of a solution of (6.1).

7 Concluding remarks

In this paper, we have established several sub-supersolution theorems. The most general one is Theorem 4.2. This theorem may be used to study a class of singular elliptic problems with convection terms. Our approach allows for the removal of the monotonicity and other technical conditions required of the singular terms in [6, 7, 17, 16, 26, 27, 29, 30] (see [23] for details). Moreover, the local Hölder continuity requirement of the convection terms, needed in [1, 10, 12, 13] may be removed. To illustrate this, we introduce two results, which have motivated our study of problems of the form (5.1).

Recently, Alves, Carrião, and Faria [1] established the following existence result for singular elliptic equations.

**Theorem 7.1** Assume that:
(a) $h : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$ are locally Hölder continuous.
(b) There exist constants $b > 0$, $0 < r_i < 1$, $(i = 1, 2, 3)$ with $r_1 < r_2$, and positive continuous functions $a_i : \Omega \to \mathbb{R}$, $(i = 1, 2, 3)$ such that
\[
b |\mu|^\gamma \leq h(x, \mu) \leq a_1(x) + a_2(x)|\mu|^{r_2} + a_3(x)|\mu|^{-r_3}, \quad \forall (x, \mu) \in \Omega \times \mathbb{R}.
\]
(c) There exist a constant $0 < r_4 < 1$, and continuous functions $a_4$ and $a_5$ such that
\[
0 \leq g(x, \eta) \leq a_4(x) + a_5(x)|\eta|^{r_4}, \quad \forall (x, \eta) \in \Omega \times \mathbb{R}^N.
\]
Then
\[
\begin{cases}
-\Delta u = h(x, u) + g(x, \nabla u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
has a solution.

Instead of employing a Galerkin method as in [1], we may apply Theorem 5.1 to obtain Theorem 7.1. Moreover, with this approach, \(h, g, \) and \(a_i, i = 1, \cdots, 5,\) are not required to be Hölder continuous.

Also, Theorem 6.1 may be used to deduce the existence result of Ghergu and Rădulescu in [13], which is stated below.

**Theorem 7.2** Let \(K < 0\) be in \(C^{0,\gamma}(\overline{\Omega}),\) \(0 < \gamma < 1,\) \(f : \overline{\Omega} \times [0, \infty) \to [0, \infty)\) be a Hölder continuous function which is positive on \(\overline{\Omega} \times (0, \infty)\) and \(g \in C^{0,\gamma}(\overline{\Omega})\) is a nonnegative and nonincreasing function.

Assume that:

(i) the mapping \((0, \infty) \ni s \mapsto \frac{f(x, s)}{s}\) is nonincreasing for all \(x \in \overline{\Omega},\)

(ii) \(\lim_{s \to 0^+} \frac{f(x, s)}{s} = \infty\) and \(\lim_{s \to \infty} \frac{f(x, s)}{s} = 0\) uniformly for \(x \in \overline{\Omega},\)

(iii) \(\lim_{s \to 0^+} g(s) = \infty.\)

Then, for \(0 \leq a \leq 2,\) the problem
\[
\begin{cases}
-\Delta u + K(x)g(u) + |\nabla u|^a = f(x, u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
has a solution.

Note that the Hölder continuity of \(f\) and \(g\) may be removed. Also, conditions (i), (ii) and (iii) may be replaced by the requirement that there exist \(0 < q < 1\) and \(c > 0\) such that
\[
|f(x, s)| \leq cs^q,
\]
for a.e. \(x \in \Omega,\) all \(s \geq 0.\)

\section{Appendix}

In this appendix we establish a result concerning a priori bounds of solutions, with respect to \(W_{0}^{1,p}(\Omega),\) for \(L^{\infty}\) bounded solutions of (2.4) under a growth condition imposed on \(f,\) suggested by a Bernstein or Nagumo growth condition used frequently in the study of nonlinear ordinary differential equations. This type of growth condition appears to have first been used by Bernstein and then extended by Nagumo (see [9], [14], [25]). The result is important in its own right, since it can be used to deduce Lemma 3.1, showing the compactness of a set of solutions to a class of nonlinear elliptic equations in the appropriate Sobolev space. The a priori bound is established using ideas from [28].

**Proposition 8.1** Consider the problem
\[
\begin{cases}
-\text{div} A(x, \nabla u) = f(x, u, \nabla u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $A$ satisfies the same conditions, as stated in Section 2, and $f$ is a Carathéodory function which satisfies the following requirement: For some positive number $M$, there exist a function $a_3 \in L^1(\Omega)$ and a constant $b_3 > 0$ such that

$$|f(x, s, \xi)| \leq a_3(x) + b_3|\xi|^p,$$

for all $s \in [-M, M]$, $\xi \in \mathbb{R}^N$, for a.e. $x \in \Omega$. Then there exists a constant $C$, depending on $M$, $a_i$, and $b_i$, $i = 1, 2, 3$, such that: If $u$ is any weak solution $u \in W^{1, p}_0(\Omega)$ of (8.1) with $|u(x)| \leq M$ for a.e. $x \in \Omega$, it follows that

$$\|u\| \leq C.$$

**Proof.** Let $M > 0$ be given and let $u$ be such a weak solution. Choose $v_t = e^{tu^2}u$, $t > 0$, as a test function for (8.1) and obtain

$$\int_{\Omega} e^{tu^2}(1 + 2tu^2)A(x, \nabla u) \cdot \nabla u dx \leq \int_{\Omega} e^{tu^2}(a_3 + b_3|\nabla u|^p)|u| dx \leq C_1 + b_3 \int_{\Omega} e^{tu^2}|\nabla u|^p|u| dx,$$

where

$$C_1 = Me^{tM^2}\int_{\Omega} a_3 dx.$$

This and condition (2.3) show that

$$\int_{\Omega} e^{tu^2}(1 + 2tu^2)(b_2|\nabla u|^p - a_2)|u| dx \leq C_1 + b_3 \int_{\Omega} e^{tu^2}|\nabla u|^p|u| dx.$$

Thus,

$$b_2 \int_{\Omega} e^{tu^2}(1 + 2tu^2)|\nabla u|^p dx \leq C_2 + b_3 \int_{\Omega} e^{tu^2}|\nabla u|^p \left(\frac{e}{2} + \frac{\mu^2}{2\epsilon}\right) dx,$$

for any $\epsilon > 0$, where

$$C_2 = C_1 + e^{tM^2}(1 + 2tM^2)\int_{\Omega} |a_2| dx.$$

We now choose $\epsilon = \frac{b_1}{4\mu^2}$ and $t$ large so that

$$b_2 - \frac{eb_3}{2} > 0$$

and

$$\int_{\Omega} e^{tu^2} \left(b_2 - \frac{eb_3}{2}\right)|\nabla u|^p dx \leq C_2.$$

The proposition follows by noting that $e^{tu^2} \geq 1$ for a.e $x \in \Omega$.

**Remark 8.1** The growth condition imposed on $f$ here is more general than condition (2.5) since $a_3$ is allowed to belong to $L^1(\Omega)$. 

References


