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A connection between cellularization for groups and spaces via two-complexes

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Abstract

Let M denote a two-dimensional Moore space (so $H_2(M; \mathbf{Z}) = 0$), with fundamental group G . The M -cellular spaces are those one can build from M by using wedges, push-outs, and telescopes (and hence all pointed homotopy colimits). The issue we address here is the characterization of the class of M -cellular spaces by means of algebraic properties derived from the group G . We show that the cellular type of the fundamental group and homological information does not suffice, and one is forced to study a certain universal extension.

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0. Introduction

Every pointed space X can be approximated by an M -cellular complex $CW_M X$ by means of a map $CW_M X \rightarrow X$ which induces a weak homotopy equivalence on pointed mapping spaces

$$\text{map}_*(M, CW_M X) \rightarrow \text{map}_*(M, X).$$

This result generalizes the well-known CW -approximation theorem of J.H.C. Whitehead (in such a case, $M = S^1$). Bousfield was the first to construct such a functor in the homotopy category, [2, Corollary 7.5]. Dror Farjoun [7] treated extensively CW_M as a functor in the category of pointed spaces and studied its general properties in the context of homotopical localization, see also [1,6]. In this paper we focus on the case when M is a two-dimensional Moore space, i.e. a two-complex with $H_2(M; \mathbf{Z}) = 0$. We study the relationship between the class of M -cellular spaces (those for which $X \simeq CW_M X$) and the group theoretical properties of $G = \pi_1 M$.

Let us denote by J the set of primes p for which G_{ab} is uniquely p -divisible. Define $R = \mathbf{Z}_{(J)}$, the integers localized at J , if G_{ab} is torsion, and $R = \bigoplus_{p \in J} \mathbf{Z}/p$ otherwise. Since M itself is HR -acyclic, so is any M -cellular

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space. We also noticed in [13] that the fundamental group of an M -cellular space is always a G -cellular group. For a group G , the class of G -cellular groups is, similarly as for spaces, the smallest class of groups containing G and closed under colimits. The group theoretical cellularization has been first studied in [13] and appears in [10] in relation with $B\mathbf{Z}/p$ -cellularization. Recently, a purely algebraical approach by Farjoun, Chachólski, Göbel, Shelah and Segev yielded important classification results [8,9].

The two observations about cellular spaces lead to a first and naive guess for the characterization of M -cellular spaces:

Question. Is a space X M -cellular if and only if $\pi_1 X$ is G -cellular and X is HR -acyclic?

The answer is yes when M is the classical Moore space $M(\mathbf{Z}/p^n, 1)$, cofiber of the degree p^n map on S^1 , [13], extending results previously obtained in [1] and [4] for Moore spaces $M(\mathbf{Z}/p^n, m)$ with $m \geq 2$. The answer is actually positive for a larger class of Moore spaces:

Theorems 2.6 and 2.11. *Let M be a two-dimensional Moore space, whose fundamental group G is either finite abelian or a subring of \mathbf{Q} . Then a space X is M -cellular if and only if $\pi_1 X$ is G -cellular and X is HR -acyclic.*

In general the class of cellular spaces does not have such a limpid characterization. There is an extra condition on the second homology group, related to the property of being “quasi- G -radical”, see Theorem 1.5. However, the characterization often holds for simply connected spaces.

Theorems 2.4 and 2.15. *Let M be a two-dimensional Moore space whose fundamental group is either torsion abelian or a subgroup of \mathbf{Q} . Then a simply connected space X is M -cellular if and only if X is HR -acyclic.*

Let us finally explain why the answer to the question is no in general. Let $G = \mathbf{Z}[1/p] * \mathbf{Z}/p$ and $M = M(\mathbf{Z}[1/p], 1) \vee M(\mathbf{Z}/p, 1)$. Then $G_{ab} = \mathbf{Z}[1/p] \oplus \mathbf{Z}/p$ and the set of primes J by which G_{ab} is uniquely divisible is the empty set. Therefore the associated ring $R = \bigoplus_{q \in J} \mathbf{Z}/q$ is 0, so the condition $\tilde{H}_*(X; R) = 0$ is always satisfied. However if the naive characterization given above was true, all simply connected spaces would be M -cellular. But this is false:

Theorem 3.2. *Let $G = \mathbf{Z}[1/p] * \mathbf{Z}/p$, and M be the Moore space $M(\mathbf{Z}[1/p], 1) \vee M(\mathbf{Z}/p, 1)$, so the associated ring R is 0. The space $K(\mathbf{Z}, 2)$ is then HR -acyclic, its fundamental group is $\mathbf{Z}[1/p] * \mathbf{Z}/p$ -cellular, but $K(\mathbf{Z}, 2)$ is not M -cellular.*

Even though this result provides a negative answer to the question we asked above, quite a few problems remain open. We ask in particular whether the naive characterization holds for Moore spaces with abelian fundamental groups. This brings us to a question which can be considered as purely group theoretical, see Question 2.2. If $M(G, 1)$ is a two-dimensional Moore space with abelian fundamental group G , is G a quotient of a subgroup of \mathbf{Q} ?

1. A general study of $M(G, 1)$ -cellular spaces

In this section M is a two-dimensional Moore space $M(G, 1)$, where $G \cong \pi_1 M(G, 1)$. The objective is to obtain a characterization of $M(G, 1)$ -cellular spaces.

1.1. Chachólski’s approach to cellularization

The fundamental theorem on which our work on cellularization relies is that of Chachólski stating that $CW_M X$ can be obtained as a certain homotopy fiber. Recall that P_M denotes the M -nullification functor, see [7]. In particular connected spaces X such that $P_M X \simeq X$ are called M -null and are characterized by the fact that $\text{map}_*(M, X) \simeq *$.

Theorem 1.1 ([6, Theorem 20.3]). *Let $X \rightarrow X'$ be a map which induces the trivial map $[M, X] \rightarrow [M, X']$ and assume that its homotopy fiber is M -cellular. Then $CW_M X$ is the homotopy fiber of the composite map $X \rightarrow X' \rightarrow P_{\Sigma M} X'$. \square*

As explained in [6, Theorem 20.5], one way to construct a map $X \rightarrow X'$ having the desired properties is the following: Take a wedge of as many copies of M as there are homotopy classes of maps $M \rightarrow X$ and consider the cofibration sequence

$$\bigvee_{[M, X]} M \xrightarrow{\varepsilon} X \rightarrow X' = P_M^1 X, \tag{1}$$

where the map ε is the evaluation map. The homotopy fiber of $X \rightarrow P_M^1 X$ is then M -cellular and $CW_M X$ is obtained as the homotopy fiber of a mixing process between the M -nullification and the ΣM -nullification. Indeed $P_M^1 X$ can be seen as the first step of the construction of $P_M X$ and $CW_M X \simeq \text{Fib}(X \rightarrow P_M^1 X \rightarrow P_{\Sigma M} P_M^1 X)$. In particular, a space X is M -cellular if and only if the space $P_M^1 X$ is ΣM -acyclic, i.e. $P_{\Sigma M} P_M^1 X \simeq *$. We show next that one can sometimes make do with less.

Corollary 1.2. *Let M be any space and assume there exists a subset I of $[M, X]$ such that the homotopy cofiber C_ε of the evaluation map $\varepsilon : \bigvee_I M \rightarrow X$ is already ΣM -acyclic. Then X is M -cellular.*

Proof. We have to show that $P_M^1 X$ is also ΣM -acyclic. For this purpose we use the cofibration sequence $\bigvee_I M \hookrightarrow \bigvee_{[M, X]} M \rightarrow \bigvee_{I'} M$, where I' is the complement of I in $[M, X]$. This induces a cofibration sequence $C_\varepsilon \rightarrow P_M^1 X \rightarrow \bigvee_{I'} \Sigma M$. But since C_ε is ΣM -acyclic, it is easy to check that so is $P_M^1 X$. \square

1.2. The characterization of $M(G, 1)$ -cellular spaces

From now on M is a two-dimensional Moore space $M(G, 1)$. The homology exact sequence associated to the cofibration (1) then has the following simple form:

$$0 \rightarrow H_2 X \rightarrow E \rightarrow \bigoplus_{[M, X]} G_{\text{ab}} \rightarrow H_1 X \rightarrow 0. \tag{2}$$

Notation 1.3. Let G be a group. As for example in [3, Section 7], J denotes the set of primes p for which G_{ab} is uniquely p -divisible, and J' its complementary set of primes. If G_{ab} is torsion, then we set $H = \bigoplus_{p \in J'} \mathbf{Z}/p$ and $R = \mathbf{Z}_{(J)}$. In the other case, i.e., if G_{ab} contains some torsion free element, $H = \mathbf{Z}[J^{-1}]$ and $R = \bigoplus_{p \in J} \mathbf{Z}/p$.

Lemma 1.4. *The map $[M, X] \rightarrow \text{Hom}(G, \pi_1 X)$ is an epimorphism for any X . \square*

Recall that the G -radical $T_G N$ of a group N is the smallest subgroup of N such that $\text{Hom}(G, N/T_G N) = 0$, [2]. When $N = T_G N$, one says that N is G -radical.

Theorem 1.5. *A space X is M -cellular if and only if the following three conditions are satisfied:*

- (1) *the fundamental group $\pi_1 X$ is G -cellular,*
- (2) *the space X is HR -acyclic,*
- (3) *the abelian group E in (2) is G_{ab} -radical.*

Proof. According to Proposition 5.3 in [13] the space $P_{\Sigma M} P_M^1 X$ is contractible if and only if $P_M^1 X$ is 1-connected, $\pi_2(P_M^1 X)$ is G_{ab} -radical and $\tilde{H}_*(P_M^1 X; R) = 0$. Since M is HR -acyclic, the cofibration (1) tells us that X is HR -acyclic if and only if $P_M^1 X$ is HR -acyclic. The associated homology exact sequence in low dimensions is (2), where $E = H_2(P_M^1 X) \cong \pi_2(P_M^1 X)$ and $P_M^1 X$ is 1-connected by the above lemma. The conclusion of the theorem is then clear. \square

Corollary 1.6. *A 1-connected space X is M -cellular if and only if X is HR -acyclic and the abelian group E in (2) is G_{ab} -radical. \square*

1.3. Quasi-radical groups

Let us remark here that in the situation of Corollary 1.6, $E \cong \text{colim}_\alpha (H_2X \hookrightarrow E_\alpha)$ where α runs over all extensions of G_{ab} by H_2X . This is a “universal” extension in the sense that any extension $H_2X \hookrightarrow E_\alpha \twoheadrightarrow G_{\text{ab}}$ can be obtained as the pull-back of it along the inclusion $i_\alpha : G_{\text{ab}} \hookrightarrow \bigoplus G_{\text{ab}}$.

Definition 1.7. Let G be any group, and let H be the group associated to G_{ab} as in Notation 1.3. An H -radical abelian group A is called *quasi- G -radical* if the universal extension

$$0 \rightarrow A \rightarrow E \longrightarrow \bigoplus_{\text{Ext}(G_{\text{ab}}, A)} G_{\text{ab}} \rightarrow 0$$

is G_{ab} -radical.

Remark 1.8. There is a strict inclusion $\{G_{\text{ab}}\text{-radical groups}\} \subseteq \{\text{quasi-}G\text{-radical groups}\}$. For $G = \mathbf{Z}_p$ for example, the group H is \mathbf{Z}/p . It is itself quasi- G -radical, but not G -radical.

Our terminology is the analogue of Mislin’s and Peschke’s notion of *h2-perfect* groups given in [12, Section 2]. In fact the two settings intersect in the most simplest cases: When $G = \mathbf{Z}/p$, an abelian group is quasi- \mathbf{Z}/p -radical if and only if it is $H\mathbf{Z}[1/p]$ 2-perfect (compare [12, Example 5.1] with Theorem 2.11) and likewise, when $G = \mathbf{Z}[1/p]$, choose $H\mathbf{Z}/p$, ordinary homology with coefficients in \mathbf{Z}/p (compare [12, Example 5.2] with Proposition 2.8).

Proposition 1.9. *Let A be an abelian group such that $K(A, 2)$ is M -acyclic. Then $K(A, 2)$ is M -cellular if and only if A is quasi- G -radical. \square*

The study of the cellularization of Eilenberg–Mac Lane spaces will be refined in Section 3.

Since the integral homology groups of an HR -acyclic space are H -radical (by the universal coefficients theorem), we obtain the following reformulation of Corollary 1.6:

Corollary 1.10. *A simply connected space X is M -cellular if and only if*

- (1) *the group π_2X is quasi- G -radical, and*
- (2) *π_kX is J' -torsion for all k , if G_{ab} is torsion, or uniquely J -divisible otherwise. \square*

1.4. Consequences for the naive characterization

It is difficult to know whether a group is quasi-radical at a first glance. Therefore it is useful to have partial answers to our question where some extra assumption on X forces π_2X to be quasi-radical. All the cases where the naive characterization holds are obtained by comparing H -radical groups with quasi- G -radical ones.

Theorem 1.11. *Let X be a space such that H_2X is G_{ab} -radical. Then X is M -cellular if and only if π_1X is G -cellular and X is HR -acyclic.*

Proof. Suppose that X is HR -acyclic, the fundamental group N is G -cellular, and H_2X is G_{ab} -radical. Let us consider a little variation in the construction of CW_MX (which in general does not produce CW_MX in the sense of Theorem 1.1). Instead of P_M^1X , we take C_ε the homotopy cofiber of the evaluation map $\bigvee_I M \rightarrow X$ where I is the union of the maps in $[M, X]$ represented by non-trivial elements in $\text{Ext}(G_{\text{ab}}, \pi_2X)$ and one preimage under $[M, X] \twoheadrightarrow \text{Hom}(G, N)$ for each morphism $G \rightarrow N$. By Corollary 1.2, X is M -cellular if we show that C_ε is ΣM -acyclic. The map $X \rightarrow K(N, 1)$ classifying the universal cover of X yields a diagram whose rows are cofibrations

$$\begin{array}{ccccc} \bigvee_I M & \rightarrow & X & \rightarrow & C_\varepsilon \\ \downarrow & & \downarrow & & \downarrow \\ \bigvee_{\text{Hom}(G, N)} M & \rightarrow & K(N, 1) & \rightarrow & K(N, 1)' \end{array}$$

Thus we get a commutative diagram in homology where the rows are exact

$$\begin{array}{ccccccc}
 0 & \rightarrow & H_2X & \rightarrow & H_2(C_\varepsilon) & \rightarrow & \bigoplus_I G_{\text{ab}} \xrightarrow{h''} H_1N \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & H_2N & \rightarrow & H_2(K(N, 1)') \xrightarrow{g} \bigoplus_{\text{Hom}(G, N)} G_{\text{ab}} \xrightarrow{h'} H_1N \rightarrow 0. & & \parallel
 \end{array}$$

The kernel of h'' is the direct sum of $\bigoplus_{\text{Ext}(G_{\text{ab}}, \pi_2 X)} G_{\text{ab}}$ and the kernel of h' . Now, since N is G -cellular by assumption, the group $H_2(K(N, 1)')$ is G_{ab} -radical by [13, Corollary 3.8], and thus $\text{Im } g = \text{Ker } h'$ is also G_{ab} -radical. Hence, $H_2(C_\varepsilon)$ is G_{ab} -radical and this proves that C_ε is ΣM -acyclic. \square

The condition of H_2X being G_{ab} -radical is not necessary. For instance, we will see in Theorem 2.12 that $X = M(\mathbf{Z}/p, 2)$ is $M(\mathbf{Z}(p^\infty), 1)$ -cellular, while \mathbf{Z}/p is not $\mathbf{Z}(p^\infty)$ -radical. It is of course quasi- $\mathbf{Z}(p^\infty)$ -radical, compared with Remark 1.8.

Corollary 1.12. *Let X be a space such that $\pi_2 X$ is G_{ab} -radical. If $\pi_1 X$ is G -cellular and X is HR -acyclic, then X is M -cellular.*

Proof. In the Hopf exact sequence $\pi_2 X \rightarrow H_2X \rightarrow H_2N$, both $\pi_2 X$ and H_2N (by [13, Corollary 2.8]) are G_{ab} -radical. Thus so is H_2X and we conclude by Theorem 1.11. \square

In fact a slight variation in the proof of Theorem 1.11 yields the following generalization of the preceding corollary.

Proposition 1.13. *Let X be a space such that $\pi_2 X$ is quasi- G -radical. Then X is M -cellular if and only if $\pi_1 X$ is G -cellular and X is HR -acyclic.* \square

2. Moore spaces with abelian fundamental group

In this section we focus on two-dimensional Moore spaces $M(G, 1)$ with abelian fundamental group G . This study will be used in the last section to give counter-examples to the naive characterization of $M(G, 1)$ -cellular spaces. Let us give concrete models for all known two-dimensional Moore spaces with abelian fundamental groups.

2.1. Varadarajan’s list

We start with the following list of all abelian groups G with $H_2(G; \mathbf{Z}) = 0$. These are exactly the groups for which a Moore space $M(G, 1)$ exists. It might however be a three-complex.

Theorem 2.1 ([14, Theorem 2.6]). *Let G be an abelian group with torsion subgroup T . For any prime p , let $T(p)$ denote the p -primary component of T . Then there exists a Moore space $M(G, 1)$ if and only if*

- (1) *the quotient G/T is of rank at most 1 over \mathbf{Q} ,*
- (2) *for all p , the group $T(p)$ is either divisible or the direct sum of a divisible group and a cyclic group,*
- (3) *for all p , we have $T(p) \otimes G/T = 0$.* \square

2.2. The torsion free case

Let S be any subgroup of \mathbf{Q} , the ring of rationals. We can assume that 1 is in S . These groups have been classified by Baer and are determined by their types (see for example [11, Theorem 85.1]). The *type* of S is the sequence $(k_2, k_3, k_5, k_7, k_{11}, \dots, k_p, \dots)$ where p runs over the set \mathcal{P} of all primes and k_p is either a natural integer or infinite. The number k_p indicates that 1 is divisible in S by p^{k_p} , but not by p^{k_p+1} , unless k_p is infinite. For example the group of type $(0, 0, \dots, 0, \infty, 0, \dots)$ is $\mathbf{Z}[1/p]$.

We now exhibit a particular construction of a Moore space $M(S, 1)$, where S is a subgroup of \mathbf{Q} of type $(k_p)_{p \in \mathcal{P}}$. Let us first order all primes and their powers up to p^{k_p} by increasing order and denote this sequence by (m_1, m_2, m_3, \dots) . Then we define α_n to be the unique prime p dividing m_n . In this way, the sequence $(\alpha_1, \alpha_2, \alpha_3, \dots)$ contains exactly k_p times the prime p . Obviously the colimit of

$$\mathbf{Z} \xrightarrow{\alpha_1} \mathbf{Z} \xrightarrow{\alpha_2} \mathbf{Z} \xrightarrow{\alpha_3} \mathbf{Z} \xrightarrow{\alpha_4} \dots$$

is S . We can also realize this at the level of spaces by a telescope of circles

$$S^1 \xrightarrow{\alpha_1} S^1 \xrightarrow{\alpha_2} S^1 \xrightarrow{\alpha_3} S^1 \xrightarrow{\alpha_4} \dots$$

and its homotopy colimit is $M(S, 1)$. It has dimension 2 since it is a telescope of one-dimensional spaces.

2.3. The torsion case

Theorem 2.1 tells us that there exist very few Moore spaces with abelian torsion fundamental groups. Such a group has to be either divisible, or a direct sum with a cyclic group. The classical Moore spaces, homotopy cofiber of the n th power map on the circle, are Moore spaces $M(\mathbf{Z}/n, 1)$ and these are two-dimensional. There exists also a two-dimensional Moore space for $\mathbf{Z}(p^\infty)$, the Prüfer group, cokernel of the canonical inclusion of \mathbf{Z} in $\mathbf{Z}[1/p]$. This is a particular case of the following construction, which uses the Moore spaces $M(S, 1)$ we defined above, where S is any subgroup of \mathbf{Q} .

Let S be a subgroup of \mathbf{Q} of type $(k_p)_{p \in \mathcal{P}}$. The set J for S consists in the primes p for which k_p is infinite, and the cokernel of the inclusion $\mathbf{Z} \hookrightarrow S$ is isomorphic to $\bigoplus_{p \in J} \mathbf{Z}(p^\infty) \oplus \bigoplus_{q \in J'} \mathbf{Z}/q^{k_q}$. Taking the corresponding element α in $\pi_1 M(S, 1)$, we get a cofibration sequence

$$S^1 \xrightarrow{\alpha} M(S, 1) \longrightarrow M\left(\bigoplus_{p \in J} \mathbf{Z}(p^\infty) \oplus \bigoplus_{q \in J'} \mathbf{Z}/q^{k_q}, 1\right).$$

This last space is again a two-complex.

Question 2.2. Are there any other two-dimensional Moore spaces with abelian fundamental groups? In particular we do not know if there exists one whose fundamental group is isomorphic to $\mathbf{Z}(p^\infty) \oplus \mathbf{Z}(p^\infty)$.

This question could be considered as purely group theoretical. The existence of a Moore space with fundamental group G is indeed equivalent to the existence of a presentation $*\mathbf{Z} \xrightarrow{\alpha} *\mathbf{Z} \rightarrow G$ such that the abelianization $\alpha_{ab} : \bigoplus \mathbf{Z} \rightarrow \bigoplus \mathbf{Z}$ is injective.

2.4. Cellularity in the non-torsion case

The set of primes J associated to a subgroup $S < \mathbf{Q}$ consists in those $p \in \mathcal{P}$ such that $k_p = \infty$. The group H is then $\mathbf{Z}[J^{-1}]$ and the ring $R = \bigoplus_J \mathbf{Z}/p$. Thus a space X is HR -acyclic if and only if its integral homology is uniquely J -divisible.

Lemma 2.3. Let M be a two-dimensional Moore space with fundamental group a subgroup S of \mathbf{Q} , of type $(k_p)_{p \in \mathcal{P}}$ and let n be an integer such that $p|n$ only if $p \notin J$. Then $M(\mathbf{Z}/n, 1)$ is M -cellular.

Proof. For such an integer n , the cokernel of the multiplication by n on S is \mathbf{Z}/n . □

Theorem 2.4. Let M be a two-dimensional Moore space with fundamental group a subgroup S of \mathbf{Q} . Then a simply connected space X is M -cellular if and only if X is HR -acyclic.

Proof. We have to prove that $S^2[J^{-1}] = M(\mathbf{Z}[J^{-1}], 2)$ is M -cellular. This is sufficient as the class of $S^2[J^{-1}]$ -cellular spaces coincides with the class of 1-connected HR -acyclic spaces.

Let us look at the inclusion $\mathbf{Z}[J^{-1}] \hookrightarrow S$. This is induced by a morphism of telescopes as follows. Define

$$\beta_k = \begin{cases} \alpha_k & \text{if } \alpha_k \in J \\ 1 & \text{if } \alpha_k \notin J \end{cases}$$

and then define by induction on k a sequence γ_k , by $\gamma_0 = 1$ and $\gamma_k = \gamma_{k-1} \cdot \beta_k$. We have then a commutative diagram

$$\begin{array}{ccccccc} \mathbf{Z} & \xrightarrow{\alpha_1} & \mathbf{Z} & \xrightarrow{\alpha_2} & \mathbf{Z} & \xrightarrow{\alpha_3} & \dots \\ \downarrow \gamma_0 & & \downarrow \gamma_1 & & \downarrow \gamma_2 & & \\ \mathbf{Z} & \xrightarrow{\beta_1} & \mathbf{Z} & \xrightarrow{\beta_2} & \mathbf{Z} & \xrightarrow{\beta_3} & \dots \end{array}$$

Replacing every copy of the integers by a copy of a circle S^1 , we get the map $S^2[J^{-1}] \rightarrow M$ as a map between telescopes. Its homotopy cofiber is thus the homotopy colimit of the homotopy cofibers of $\gamma_k : S^1 \rightarrow S^1$. These are Moore spaces of type $M(\mathbf{Z}/\gamma_k, 1)$. Notice that no prime $p \in J$ divides any γ_k , i.e. γ_k satisfies the conditions of Lemma 2.3. Hence $M(\mathbf{Z}/\gamma_k, 1)$ is M -cellular and their telescope as well: $M(\bigoplus_{p \notin J} \mathbf{Z}/p^{k_p}, 1)$ is M -cellular. The Puppe sequence tells us finally that we have a cofibration

$$M \longrightarrow M(\bigoplus_{p \notin J} \mathbf{Z}/p^{k_p}, 1) \longrightarrow S^2[J^{-1}]$$

where the first two spaces are M -cellular. Thus so is the third and we are done. \square

Example 2.5. Let S be the subgroup of \mathbf{Q} of type $(1, 1, 1, 1, \dots)$, that is, S is the additive subgroup of \mathbf{Q} generated by $1/p$ for all prime numbers p . The above theorem shows in this case that S^2 is $M(S, 1)$ -cellular. There is an isomorphism $\text{Ext}(S, \mathbf{Z}) \cong (\prod_p \mathbf{Z}/p)/\mathbf{Z}$ and there are two possibilities for an extension $\mathbf{Z} \rightarrow E \rightarrow S$. Either the S -reduction of E (the quotient by its S -radical) is E , or it is \mathbf{Z} , depending on whether the element $\alpha \in (\prod_p \mathbf{Z}/p)/\mathbf{Z}$ representing the extension is torsion or not. Thus none of these extensions are S -radical. However, if we take $F = \text{colim}_{E \in \text{Ext}(S, \mathbf{Z})} (\mathbf{Z} \hookrightarrow E)$, then F is S -radical (i.e. \mathbf{Z} is S -quasi-radical).

2.5. The case of subrings of \mathbf{Q}

We have here a stronger result than Theorem 2.4 and we can in fact completely determine the class of M -cellular spaces.

Theorem 2.6. Let $M = M(\mathbf{Z}[J^{-1}], 1)$ be a two-dimensional Moore space. Then a space X is M -cellular if and only if $\pi_1 X$ is $\mathbf{Z}[J^{-1}]$ -cellular and X is HR -acyclic.

Proof. This result is a reflection of the fact that the group H associated to $\mathbf{Z}[J^{-1}]$ as introduced in Notation 1.3 is $\mathbf{Z}[J^{-1}]$ itself. This means we can directly apply Theorem 1.11 because $H_2(X; \mathbf{Z})$ is H -radical. \square

Lemma 2.7. Let $G = \mathbf{Z}[J^{-1}]$ and $M = M(G, 1)$ be a two-dimensional Moore space. Then the class of nilpotent M -cellular spaces coincides with that of nilpotent M -acyclic spaces.

Proof. An M -cellular space is always M -acyclic, Theorem 1.1. Thus let X be an M -acyclic space. We apply Chachólski’s theorem to prove that it is also M -cellular. The cofiber $P_M^1 X$ of the evaluation $\vee M \rightarrow X$ is a simply connected space because there is no difference between G -socular and G -radical nilpotent groups (there is none for abelian groups and use [3, Proposition 7.4]). It is also M -acyclic and hence $H_n(P_M^1 X; \mathbf{Z})$ has to be uniquely J -divisible for all $n \geq 2$, i.e. G -radical. Therefore $P_{\Sigma M} P_M^1 X \simeq *$ and we are done. \square

Proposition 2.8. Let $G = \mathbf{Z}[J^{-1}]$ and $M = M(G, 1)$ be a two-dimensional Moore space. Then, for any nilpotent space X with G -radical fundamental group, we have $CW_M X \simeq \text{Fib}(X \rightarrow \prod_J \widehat{X}_p)$.

Proof. This is a direct consequence of the preceding lemma and [5, Theorem 4.4], which identifies $P_M X$ with $\prod_J \widehat{X}_p$ when X is a nilpotent space with G -radical fundamental group. \square

Example 2.9. Let $G = \mathbf{Q}$. Then $CW_M S^2$ is a two-stage Postnikov space, with

$$\pi_1 CW_M S^2 \cong \pi_2 CW_M S^2 \cong \left(\prod_{\mathcal{P}} \widehat{\mathbf{Z}}_p \right) / \mathbf{Z} \cong \bigoplus \mathbf{Q}.$$

This gives in particular an example where the fundamental group of the M -cellularization drastically differs from the G -cellularization of the fundamental group.

Example 2.10. Let $G = \mathbf{Z}[1/p]$. Then $CW_M K(\mathbf{Z}(p^\infty), 1) \simeq K(\widehat{\mathbf{Q}}_p, 1)$ and the associated universal central extension (as in [13, Theorem 2.7]) is the extension $\widehat{\mathbf{Z}}_p \rightarrow \widehat{\mathbf{Q}}_p \rightarrow \mathbf{Z}(p^\infty)$.

2.6. Cellularity in the reduced torsion case

The naive description of the class of $M(G, 1)$ -cellular spaces given in the introduction is an actual characterization when G is any finite cyclic group. More generally we have:

Theorem 2.11. *Let $G = \bigoplus \mathbf{Z}/p^{k_p}$ be an abelian torsion group with no divisible summand and $M = M(G, 1)$ be a two-dimensional Moore space. Then a space X is M -cellular if and only if $\pi_1 X$ is generated by elements of order p^l for $l \leq k_p$ and X is $H\mathbf{Z}_{(J)}$ -acyclic.*

Proof. The hypothesis on the fundamental group tells us that the cofiber X' in Theorem 1.1 is simply connected. Moreover, since X is $H\mathbf{Z}_{(J)}$ -acyclic, so is X' . We conclude by Bousfield’s explicit computations in [3, Theorem 7.5] that $P_{\Sigma M} X'$ is contractible. \square

For cyclic groups of prime power order, we recover [13, Theorem 6.2].

2.7. Cellularity in the unreduced torsion case

When $G = \mathbf{Z}(p^\infty)$, the same characterization holds in the simply connected case.

Theorem 2.12. *Let $M = M(\mathbf{Z}(p^\infty), 1)$. Then a 1-connected space X is M -cellular if and only if X is $H\mathbf{Z}[1/p]$ -acyclic.*

Proof. The space $M(\mathbf{Z}/p^k, 2)$ is the homotopy cofiber of the map $M \rightarrow M$ induced by the multiplication by p^k on $\mathbf{Z}(p^\infty)$, so it is M -cellular. Hence any simply connected p -torsion space is also so (see [4]). \square

We will need the following lemma to analyze the class of $M(G, 1)$ -cellular spaces for a general abelian torsion group G .

Lemma 2.13. *Let A and B be two abelian groups such that there exist a two-dimensional Moore space M of type $M(A \oplus B, 1)$. Then both $M(A, 1)$ and $M(B, 1)$ are M -cellular.*

Proof. Let $M \rightarrow M(A, 1)$ be any map which induces the projection $A \oplus B \rightarrow A$ on the fundamental group. Its homotopy cofiber is $M(B, 2)$, which is ΣM -acyclic. Thus $M(A, 1)$ is M -cellular by Chachólski’s theorem, or rather by its Corollary 1.2. \square

Remark 2.14. We point out that it is not clear whether $M(A, 1)$ is a retract of $M(A \oplus B, 1)$, which would provide a direct proof of the above lemma. One can of course construct maps $M(A, 1) \rightarrow M(A \oplus B, 1)$ and $M(A \oplus B, 1) \rightarrow M(A, 1)$ which induce the canonical inclusion and projection, but the composite might fail to be the identity, compare with Lemma 1.4.

We can now prove the same result as Theorem 2.12 for any $M(G, 1)$ with G -torsion and abelian. Together with Theorem 2.4, it shows that the naive characterization given in the introduction holds for all Moore spaces constructed at the beginning of this section, at least in the 1-connected case.

Theorem 2.15. *Let $M = M(G, 1)$ be a two-dimensional Moore space with torsion abelian fundamental group G . Then a 1-connected space X is M -cellular if and only if X is $H\mathbf{Z}_{(J)}$ -acyclic.*

Proof. By Theorem 2.1 we infer that $T(p)$ is either cyclic, or the direct sum of a divisible group and a cyclic one, for all $p \in J$. Since $\text{Ext}(T(p), T(q))$ is zero, our group G decomposes as a direct sum $\bigoplus T(p)$ and each of the components contains either a copy of \mathbf{Z}/p^k or one of $\mathbf{Z}(p^\infty)$ as a direct summand. This implies by Lemma 2.13 that either $M(\mathbf{Z}/p^k, 1)$ or $M(\mathbf{Z}(p^\infty), 1)$ is M -cellular. We conclude now by Theorems 2.11 and 2.12. \square

3. Counter-examples to the naive characterization

In this section we show that the naive description of cellular spaces fails, even in the simply connected case. It is therefore necessary in general to work with the more complicated characterization of Theorem 1.5 and the notion of quasi-radical groups. Our counter-example relies on the following computation.

3.1. The cellularization of $K(A, 2)$

We compute here the $M(G, 1)$ -cellularization of an Eilenberg–Mac Lane space $K(A, 2)$, where A is any abelian group. For simplicity we shall suppose that $K(A, 2)$ is HR -acyclic. This is no longer a restriction, since for calculating $CW_M K(A, 2)$ we can first calculate $\bar{P}_M K(A, 2)$, which is a certain product $K(A', 2) \times K(N', 1)$ by [13, Proposition 6.1] and then apply CW_M . Note that $K(A', 2)$ is HR -acyclic and it is easy to calculate $CW_M K(N', 1)$.

Theorem 3.1. *Let $M = M(G, 1)$ be a two-dimensional Moore space and R be the ring associated to G . Suppose that $K(A, 2)$ is HR -acyclic. Then,*

$$CW_M K(A, 2) \simeq K(\text{Ker } \varphi, 2) \times K(\text{Coker } \varphi, 1)$$

where $\varphi : A \rightarrow E \rightarrow E/T_G E$, and E is the “universal” extension for A as in Definition 1.7.

Proof. Let us consider the map $K(A, 2) \rightarrow K(E, 2)$. It satisfies the conditions of Theorem 1.1 since its homotopy fiber is $K(\oplus_{\text{ab}} G, 1)$, an M -cellular space by Corollary 1.12. We must therefore compute $P_{\Sigma M} K(E, 2)$ to identify the cellularization of $K(A, 2)$.

We use now Bousfield’s formulas in [3, Theorem 7.5]. When G_{ab} is an abelian torsion group $P_{\Sigma M} K(E, 2) \simeq K(E/T_G E, 2)$. When G_{ab} is not a torsion group, then $P_{\Sigma M} K(E, 2) \simeq K(E/T_G E, 2) \times K(B, 3)$ where $B = \prod_J \text{Hom}(\mathbf{Z}_{p^\infty}, E)$. But $K(E, 2)$ is HR -acyclic, $R = \oplus_J \mathbf{Z}/p$ here, so $B = 0$. Hence, in both cases, $CW_M K(A, 2)$ is the homotopy fiber of the map $K(A, 2) \rightarrow K(E/T_G E, 2)$. \square

3.2. Failure of the naive characterization

Let $G = \mathbf{Z}[1/p] * \mathbf{Z}/p$ and $M = M(\mathbf{Z}[1/p], 1) \vee M(\mathbf{Z}/p, 1)$. Then $G_{\text{ab}} = \mathbf{Z}[1/p] \oplus \mathbf{Z}/p$ is not torsion, the associated set of primes J as introduced in Notation 1.3 is the empty set, $H = \mathbf{Z}$, and $R = 0$, so that all spaces are HR -acyclic. If the naive characterization given in the introduction was true, all simply connected spaces would be M -cellular. But this is false:

Theorem 3.2. *Let $G = \mathbf{Z}[1/p] * \mathbf{Z}/p$, and M be the Moore space $M(\mathbf{Z}[1/p], 1) \vee M(\mathbf{Z}/p, 1)$, so the associated ring R is 0. The space $K(\mathbf{Z}, 2)$ is then HR -acyclic, its fundamental group is $\mathbf{Z}[1/p] * \mathbf{Z}/p$ -cellular, but $K(\mathbf{Z}, 2)$ is not M -cellular.*

Proof. Let us compute explicitly $CW_M K(\mathbf{Z}, 2)$. We consider the composite map $\alpha : \mathbf{Z} \xrightarrow{p} \mathbf{Z} \rightarrow \mathbf{Z}_p^\wedge$, or equivalently $\mathbf{Z} \rightarrow \mathbf{Z}_p^\wedge \xrightarrow{p} \mathbf{Z}_p^\wedge$, where both morphisms to the p -adic integers are the completion maps. The short exact sequence of cokernels shows that the cokernel of the composite is $\mathbf{Z}_p^\wedge/\mathbf{Z} \times \mathbf{Z}/p$. Consider the fibration

$$K(\mathbf{Z}_p^\wedge/\mathbf{Z} \times \mathbf{Z}/p, 1) \longrightarrow K(\mathbf{Z}, 2) \xrightarrow{\alpha} K(\mathbf{Z}_p^\wedge, 2).$$

Since $K(\mathbf{Z}_p^\wedge/\mathbf{Z}, 1) \simeq CW_{M(\mathbf{Z}[1/p], 1)} K(\mathbf{Z}, 2)$ by Proposition 2.8, it is M -cellular, and so is $K(\mathbf{Z}/p, 1) \simeq CW_{M(\mathbf{Z}/p, 1)} K(\mathbf{Z}, 2)$. The map α satisfies thus the conditions of Theorem 1.1. But notice now that the space $K(\mathbf{Z}_p^\wedge, 2)$ is ΣM -null, which exhibits $CW_M K(\mathbf{Z}, 2)$ as $K(\mathbf{Z}_p^\wedge/\mathbf{Z} \times \mathbf{Z}/p, 1)$. In particular $K(\mathbf{Z}, 2)$ is not M -cellular. \square

The Moore space in the above counter-example has a non-abelian fundamental group. We do not know whether the naive characterization holds in the abelian case.

Question 3.3. Let $M = M(G, 1)$ be a two-dimensional Moore space with G abelian. Is it true that a space X is M -cellular if and only if $\pi_1 X$ is G -cellular and X is HR -acyclic? The question is even open for $G = \mathbf{Z}(p^\infty)$.

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