Decomposing random graphs into few cycles and edges

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joint work with Michael Krivelevich and Benny Sudakov
Path decompositions

Gallai's conjecture: Every connected graph on \( n \) vertices can be decomposed into \( \lfloor \frac{n + 1}{2} \rfloor \) paths.

Theorem (Lovász, 1968): Every graph on \( n \) vertices can be decomposed into at most \( \frac{n}{2} \) cycles and paths. Hence also into at most \( n \) paths.

Theorem (Yan, 1999; Dean–Kouider, 2000): Every graph on \( n \) vertices can be decomposed into at most \( \frac{2n}{3} \) paths.
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Cycle decompositions

Erdős–Gallai conjecture

The edge set of every graph on \( n \) vertices can be decomposed into \( O(n) \) cycles and edges.

Claim (folklore)

Every graph can be decomposed into \( O(n \log n) \) cycles and edges.

Proof.

A graph of average degree \( d \) contains a cycle of length at least \( d \).

Dropping to average degree \( d/2 \) takes at most \( n \) cycles.

After removing \( O(n \log n) \) cycles, a forest remains.
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Theorem (Conlon–Fox–Sudakov, 2013+)

> Every graph breaks up into $O(n \log \log n)$ cycles and edges.
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- The conjecture holds for random graphs and graphs of linear minimum degree.
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- Every graph breaks up into $O(n \log \log n)$ cycles and edges.
- The conjecture holds for random graphs and graphs of linear minimum degree.

Our result addresses the random graph bound and determines the right asymptotics.
Random graphs

Definition

The Erdős-Rényi random graph $G(n, p)$ is a random subgraph of $K_n$, where the edges are kept independently with probability $p$.

Definition

Let $p = p(n)$ be some probability function. We say that some property $P$ holds for $G(n, p)$ with high probability or whp, if

$$\lim_{n \to \infty} P(\text{$P$ holds for $G(n, p)$}) = 1.$$
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$$\lim_{n \to \infty} \mathbb{P}(P \text{ holds for } G(n, p)) = 1.$$
Some natural lower bounds

Let $\text{odd}(G)$ be the number of odd-degree vertices in $G$. Each such vertex needs to be the endpoint of an edge. We need at least $\frac{\text{odd}(G(n,p))}{2}$ edges. $G(n,p)$ has about $\binom{n}{2}p$ edges whp. A cycle may contain up to $n$ edges. We need at least $np/2$ cycles. Altogether, at least $\text{odd}(G(n,p))^2 + np^2$ cycles and edges.
Some natural lower bounds

Let $odd(G)$ be the number of odd-degree vertices in $G$. 

Each such vertex needs to be the endpoint of an edge.

We need at least $\frac{odd(G)}{2}$ edges. $G(n, p)$ has about $\frac{n^2p}{2}$ edges whp. 

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Altogether, at least $odd(G) + np/2$ cycles and edges.
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Altogether, at least $\frac{odd(G(n,p))}{2} + \frac{np}{2}$ cycles and edges.
Our result

**Theorem (K–Krivelevich–Sudakov, 2014+)**

If \( p \gg \frac{\log \log n}{n} \) then whp, \( G(n, p) \) can be decomposed into

\[
\text{odd}(G(n, p)) + \frac{np}{2} + o(n)
\]

*cycles and edges.*
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$$\frac{\text{odd}(G(n, p))}{2} + \frac{np}{2} + o(n)$$

cycles and edges.

Remark. In most of the probability range, $\text{odd}(G(n, p)) \sim n/2$. 
Sparse random graphs \( \left( \frac{\log n}{n} \ll p \leq \frac{\log^{10} n}{n} \right) \)

We need to show odd \((G(n, p))\) 2 + o\((n)\) cycles and edges are enough.

Plan:
1. Remove edges to obtain an Euler graph.
2. Then remove long cycles to get an Euler graph on linearly many edges.
3. Break it up into cycles arbitrarily.
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The odd-degree vertices

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\[ \alpha(G(n,p)) \leq 2 \log(np) \]

\[ \text{diam}(G(n,p)) \leq 2 \log(n \log(np)) \]

For \( p \gg \log n \) the product is \( 4 \log n p = o(n) \).
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- $\alpha(G(n, p)) \leq \frac{2 \log(np)}{p}$
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Two facts

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For \( p \gg \frac{\log n}{n} \) the product is \( \frac{4 \log n}{p} = o(n) \).
Aim: Given a subgraph of $G(n, p)$ of average degree $d$, find a cycle of length $d \log^2 n$. 

Recall: When removing cycles of length $d$, $n$ cycles were needed to halve the average degree.

Here: We remove cycles of length $d \log^2 n$, so $n \log^2 n$ cycles are enough to halve the average degree.
Finding long cycles

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We remove cycles of length $d \log^2 n$, so $\frac{n}{\log^2 n}$ cycles are enough to halve the average degree.
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Aim: Find a cycle of length $d \log^2 n$. 

A Posá-type lemma

If $G$ has no cycle of length at least $3t$ then there is a set $T$ of size $t$ with $|N(T)| \leq 2|T|$. 

$S$ is too dense to be a subgraph of $G(n, p)$. 

$\deg \approx d \frac{1}{6} d^2 \log^2 n$ edges 

Density $\approx \frac{1}{\log^2 n}$.
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T & \quad N(T)
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\begin{figure}
\centering
\begin{tikzpicture}
\node (S) at (0,0) {$S$};
\node (T) at (-1,-1) {$T$};
\node (N_T) at (1,-1) {$N(T)$};
\draw (S) edge (T);
\end{tikzpicture}
\end{figure}
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\begin{align*}
S & \quad \text{\text{d log}^2 n \text{ vertices}} \\
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\[ S \quad \text{\(d \log^2 n\) vertices} \]

\[ d \approx d \quad \text{\(d \log^2 n\) vertices} \]

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$N(T)$

$\frac{1}{6} d^2 \log^2 n$ edges
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We are left with an Euler graph with $O(n)$ edges.
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Claim
$G(n, p)$ contains at most $\sqrt{n}$ cycles of length at most $\log \log n$. 
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Claim

$G(n, p)$ contains at most $\sqrt{n}$ cycles of length at most $\log \log \log n$.

$O(n / \log \log n)$ cycles in total.
Dense random graphs

- Remove odd (\(G(n,p)\)) / 2 + o(n) edges to make the graph Euler
- Remove approximately \(np/2\) Hamilton cycles using a result of Knox–Kühn–Osthus.
- Then a miracle occurs..
- Hence the graph is sparse enough to use our previous tools.
Dense random graphs

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“I think you should be more explicit here in step two.”
What kind of miracle?
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- Break the remaining edges into matchings
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- Break the remaining edges into matchings
- Use the random structure to connect them into cycles

Broder–Frieze–Suen–Upfal give a criterion when a set of vertex pairs can be connected by vertex-disjoint paths in a random graph.

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Open problems

1. What happens for very small $p$? ($p = O(\log \log n / n)$)

$\triangleright$ How many edges need to be removed to make $G(n, p)$ Euler?

2. The Erdős–Gallai conjecture: Can any graph on $n$ vertices be decomposed into $O(n)$ cycles and edges?
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   ▶ How many edges need to be removed to make $G(n, p)$ Euler?

2. The Erdős–Gallai conjecture:
   Can any graph on $n$ vertices be decomposed into $O(n)$ cycles and edges?
Thank you!