Exercise sheet 11: Extremal graph theory

1. Prove that if a graph has $m$ edges, then it has a bipartite subgraph with $\left\lceil \frac{m}{2} \right\rceil$ edges.

2. Prove that if a graph on $n$ vertices contains no even cycle, then it has less than $2n$ edges. Is this tight?

3. Let $v_1, \ldots, v_n \in \mathbb{R}^d$ be vectors of length at least 1. Show that the number of pairs $v_i, v_j$ with $\|v_i + v_j\| < 1$ is at most $\left\lfloor \frac{n^2}{4} \right\rfloor$.

4. Show that a graph on $n \geq 4$ vertices with $2n - 3$ edges contains a cycle with a chord.

5. If $G$ is a non-bipartite graph with $n$ vertices and more than $(n - 1)^2/4 + 1$ edges, show that it contains a triangle.

6. Show that a graph on $n$ vertices with $n + 4$ edges must contain two edge-disjoint cycles.

Bonus Problem: Little Nicholas decided to become a great writer. To that end, he devised a serious working plan. From the beginning he opted to write in French, for that is the language of love. To each letter of the French alphabet (of which there are 26), he carefully assigned a charming word which contains that letter. As a wise person, Little Nicholas never uses diacritics, as those serve for nothing, and besides, nobody ever knows where to put one. To compose his first novel he proceeded like this. First he wrote down only the word assigned to the letter “a”. Then he replaced every letter in that word with the word assigned to it, separating the words with a space. To the resulting sentence he applied the same trick of replacing the letters with their corresponding words, and so on, repeating the replacement procedure 40 times in total. And voilà, the masterpiece was born! From the very first words one could smell a true cliffhanger: “Cent sangsues sans sang sucent son sang”. Prove that this remarkable sentence reappears in Nicholas’ novel.

This week’s bonus problem is non-submittable (and non-refundable).

If you spot any mistakes on this sheet, please drop an email to filip.moric@epfl.ch.

Given out: Thursday, May 23

Due: ♥♥♥