Exercise sheet 4: Connectivity

1. What is the largest number of edges that a graph on \( n \) vertices can have without being connected?

2. Show that \( \kappa(G) \leq \kappa'(G) \leq \delta(G) \) for any graph \( G \).

3. Let \( G \) be a \( k \)-regular bipartite graph with \( k \geq 2 \). Show that \( G \) has no cut edge.

4. Show that any two longest cycles in a connected graph without cut vertices have at least two vertices in common.

5. Show that if \( G \) is a graph, with \( |V(G)| = n \geq k + 1 \) and \( \delta(G) \geq (n + k - 2)/2 \), then \( G \) is \( k \)-connected.

6. Using Menger’s theorem, prove the following so called Fan lemma: Let \( G \) be a \( k \)-connected graph, let \( x \) be a vertex of \( G \), and let \( Y \subseteq V \setminus \{x\} \) be a set of at least \( k \) vertices of \( G \). Then there exists a \( k \)-fan in \( G \) from \( x \) to \( Y \) (that is, a family of \( k \) internally disjoint \((x,Y)\)-paths whose terminal vertices are distinct).

7. Let \( k \geq 2 \). Show that every \( k \)-connected graph of order at least \( 2k \) contains a cycle of length at least \( 2k \).

8. Show that a 3-connected non-bipartite graph contains at least four odd cycles.

9. (extra exercise for last week) Prove Dilworth’s theorem using the Gallai-Milgram theorem. NB: Dilworth’s theorem says that in any finite poset, the maximum number of elements in an antichain equals the minimum number of chains in any partition of the poset into chains.

Bonus Problem: There are \( N \) cities in Graphland. Between some pairs of cities there are non-stop (direct) connections, operated by some of the local companies (a connection provides flights in both directions). It turned out that for any \( k \) (\( 2 \leq k \leq N \)) and for any choice of \( k \) cities, the number of connections between these cities does not exceed \( 2k - 2 \). Prove that all the connections can be divided between two airlines, Graphair and GraphyJet, in such a way that there will be no cyclic air trip, in which all flights are served by the same company.

- The assignment is due on Thursday, March 21 at the exercise session
- Submit a solution to the bonus problem only
- Please typeset your homework (le français is acceptable)

If you spot any mistakes on this sheet, please drop an email to filip.moric@epfl.ch.