Exercise sheet 6: Solutions

Caveat emptor: These are merely extended hints, rather than complete solutions.

1. What is the maximum number of edges that a graph on \( n \) vertices can have without containing a subdivision of \( K_3 \)?

   **Solution.** “A subdivision of \( K_3 \)” is a fancy way to say “a cycle”. Hence the answer is \( n - 1 \).

2. If \( G \) is a graph such that either \( |E(G)| < 9 \) or \( |V(G)| < 5 \), prove that \( G \) is planar.

   **Solution.** In order to contain a subdivision of \( K_{3,3} \), a graph must have at least 9 edges and at least 6 vertices, while a subdivision of \( K_5 \) requires at least 10 edges and at least 5 vertices.

3. If \( x, y, z \) are distinct vertices of a planar graph on \( n \) vertices, prove that

   \[
   d(x) + d(y) + d(z) \leq 2n + 2.
   \]

   **Solution.** Consider a multiset whose elements are all neighbors of \( x \), then all neighbors of \( y \), and finally, all neighbors of \( z \). Since \( K_{3,3} \) is forbidden, each vertex appears at most twice in the multiset, except perhaps for two vertices that can appear 3 times. Thus the total size of the multiset is \( \leq 2(n - 2) + 6 = 2n + 2 \).

4. The edges of \( K_{11} \) are colored in two colors: red and blue. Prove that either the red graph, or the blue graph, is not planar.

   **Solution.** A planar graph on 11 vertices has at most \( 3 \cdot 11 - 6 = 27 \) edges. The number of edges of \( K_{11} \) is \( 11 \cdot 5 = 55 > 2 \cdot 27 \).

5. Prove that a graph is embeddable on the sphere if and only if it is planar.

   **Solution.** Use stereographic projection.
6. Prove that any convex polytope in \( \mathbb{R}^3 \), whose vertices are pairwise adjacent, is a tetrahedron.

**Solution.** A graph of a convex polytope in \( \mathbb{R}^3 \) is always a planar graph, so it cannot contain \( K_5 \) as a subgraph.

7. Prove that any convex polytope in \( \mathbb{R}^3 \) contains either a triangular face, or a vertex incident to exactly three edges.

**Solution.** Suppose neither of the two things happens. Since \( \sum d(v) = 2e \), we have that \( 2e \geq 4n \). On the other hand, Euler’s formula gives \( 2e = \sum d(f) \geq 4(e - n + 2) \), which implies \( e \leq 2n - 4 \). Contradiction.

8. Is there a 3-dimensional convex polytope with exactly 7 edges?

**Solution.** If \( e = 7 \), then \( n + f = 9 \), by Euler’s formula. Any 3-dimensional convex polytope has at least 4 vertices and at least 4 facets. Moreover, if it has either 4 vertices or 4 facets, then it must be a tetrahedron. In our case, \( n + f = 9 \) implies that either \( n = 4 \) and \( f = 5 \), or \( n = 5 \) and \( f = 4 \). None of the cases is possible.

9. Prove that the graph of a 3-dimensional convex polytope is always 3-connected.

**Solution.** We need to prove that if we delete any two vertices \( u \) and \( v \) of \( P \), the graph will remain connected. Consider a plane \( \pi \) that contains \( u \), \( v \) and another vertex of the polytope, say \( w \). Assume that there are vertices of \( P \) on both sides of \( \pi \) (the other case is similar). By moving \( \pi \) parallel to itself as long as the intersection with the polytope is non-empty, we’ll have two extremal positions \( \pi_1 \) and \( \pi_2 \), when the intersection is a face of \( P \). Call the two faces \( F_1 \) and \( F_2 \). Now for all vertices that are between \( \pi_1 \) and \( \pi \) there is a “downward” path (always progressing towards \( \pi_1 \)) that connects them to a vertex of \( F_1 \), and \( F_1 \) is connected itself. The same reasoning applies on the other side, and it remains to notice that the vertex \( w \) belongs to both sides.

10. Show that a graph is outerplanar if and only if it does not contain a subdivision of \( K_4 \) or \( K_{2,3} \).

**Solution.** For the direction “\( \Rightarrow \)” it suffices to prove that neither \( K_4 \) nor \( K_{2,3} \) are outerplanar. This is easy, since if they had an outerplanar drawing it could be turned into a planar embedding of \( K_5 \) or \( K_{3,3} \), by
adding a vertex and the corresponding edges in the outer face. To prove the converse, suppose $G$ doesn’t contain a subdivision of $K_4$ or $K_{2,3}$. Then the graph $G'$, defined by $V(G') = V(G) \cup \{x\}$ and $E(G') = E(G) \cup \{xv : v \in V(G)\}$, doesn’t contain a subdivision of $K_5$ or $K_{3,3}$. Thus, by Kuratowski’s theorem $G'$ is planar, which easily implies that $G$ is outerplanar.

11. Out of pure boredom, so typical for his age, Little Nicholas drew a closed self-intersecting curve in the plane. The curve divided the plane into simply connected regions. Prove that he can color these regions with two colors, red and blue, such that no two regions that share a boundary arc are of the same color. (Since Little Nicholas is just a human, you can assume that the curve intersects itself only finitely many times.)

Solution. We can see Nicholas’ picture as a drawing of a naturally defined planar graph. We want to show that its dual is bipartite. We need to show that the dual doesn’t contain an odd cycle. A cycle in the dual corresponds to a simple closed curve, where the length of the cycle is equal to the number of intersection points between the cycle-curve and Nicholas’ curve. However, this number is clearly even, since by Jordan’s theorem the cycle-curve has its interior and its exterior, and while traversing Nicholas’ curve we always change the side.

12. Pinocchio got a present: it’s a cute little convex polytope with 10000 faces. Prove that his present has at least 1000 faces with the same number of sides.

Solution. Let $f, e, n$ be the number of faces, edges and vertices of the polytope, respectively. Denote by $f_i$ the number of faces of length $i$, for all $i \geq 3$. The inequality $2e \geq 3n$, together with Euler’s formula, implies that $e \leq 3f - 6$, which translates into

$$3f_3 + 4f_4 + 5f_5 + \cdots \leq (6f_3 + 6f_4 + 6f_5 + \cdots) - 12,$$

or

$$4f_3 + 3f_4 + 2f_5 + f_6 \geq f + 12.$$

It follows that one of the numbers $f_3, f_4, f_5, f_6$ is bigger than $f/10$.

If you spot any mistakes on this sheet, please drop an email to filip.moric@epfl.ch.