Exercise sheet 9: Solutions

Caveat emptor: These are merely extended hints, rather than complete solutions.

1. Show that $\chi(G) \leq 1 + \max\{\delta(F) : F \subseteq G\}$, for every graph $G$.

Solution. By induction on the number of vertices. Take out a vertex $v$ of minimum degree and apply induction to $G - v$.

2. Show that $\chi(G)\chi(\overline{G}) \geq |V(G)|$, for every graph $G$.

Solution. Consider two colorings $c_1$ and $c_2$ of $G$ and $\overline{G}$ with $\chi(G)$ and $\chi(\overline{G})$ colors respectively. Then $c(v) = (c_1(v), c_2(v))$ is a proper coloring of the complete graph $G \cup \overline{G}$ with $\chi(G)\chi(\overline{G})$ colors.

3. Show that, for any graph $G$ with $n$ vertices and $m$ edges, we have

$$\chi(G) \geq \frac{n^2}{n^2 - 2m}.$$ 

Solution. We have to show that if a graph is $k$-partite, then $k \geq \frac{n^2}{n^2 - 2m}$, or equivalently, $m \leq \frac{k-1}{2k}n^2$. A $k$-partite graph will have the largest possible number of edges if it’s complete and the sizes of its parts are as equal as possible (since by replacing classes with $x$ and $y$ vertices by two classes with $\lceil \frac{x+y}{2} \rceil$ and $\lfloor \frac{x+y}{2} \rfloor$ vertices, the total number of edges won’t decrease).

4. Determine the chromatic number of the line graph of $K_n$.

Solution. Every color class is a matching in $K_n$, and thus every color class contains at most $\lfloor \frac{n}{2} \rfloor$ edges. This gives that $\chi(L(K_n)) \geq n - 1$ if $n$ is even, and $\chi(L(K_n)) \geq n$ if $n$ is odd. In both cases we have equality. For $n$ odd, just draw $K_n$ as a regular $n$-gon, and color parallel edges with the same color. If $n$ is even, then we use an optimal coloring of $L(K_{n-1})$ and upgrade it to a coloring of $L(K_n)$.

5. (a) If $\mathcal{F}$ is a family of closed intervals in $\mathbb{R}$ any two of which have a non-empty intersection, then all intervals in $\mathcal{F}$ have a non-empty intersection.
(b) Prove that an interval graph is chordal.

(c) If $\mathcal{F}$ is a finite family of closed intervals in $\mathbb{R}$, such that no point is covered by more than $k$ intervals, then the intervals in $\mathcal{F}$ can be colored with $k$ colors, so that no two intervals of the same color intersect.

(d) If $\mathcal{F}$ is a finite family of closed intervals in $\mathbb{R}$, no more than $k$ of which are pairwise disjoint, then the intervals in $\mathcal{F}$ can be colored with $k$ colors, so that no two intervals of the same color are disjoint.

Solution. (a) For any interval $I \in \mathcal{F}$ define $\ell(I)$ and $r(I)$ to be its left and right endpoint, respectively. Let $m = \sup_{I \in \mathcal{F}} \ell(I)$ and $M = \inf_{I \in \mathcal{F}} r(I)$. Then $m \leq M$, since otherwise there would be two disjoint intervals.

(b) Suppose the contrary and let $[a_1, b_1], \ldots, [a_k, b_k]$ be intervals representing an induced cycle of length $k \geq 4$. W.l.o.g. we can assume that $a_1 < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < \ldots$, but then the last interval won’t intersect the first one. Contradiction.

(c) We know that interval graphs are chordal, and thus perfect. Then use (a).

(d) The complement is also perfect.

6. Show that $K_{m,n}$ is $k$-list-colorable for all $k \geq \min\{m, n\} + 1$.

Solution. By induction on $\min\{m, n\}$. Take one vertex from the part with a smaller number of vertices and assign to it any color $c$ from its list. Then in the rest of the graph, remove the color $c$ from all lists where it appears. The rest is still list-colorable by induction.

7. Show that if $m = \binom{2k-1}{k}$, then $K_{m,m}$ is not $k$-list-colorable.

Solution. Assign to all the vertices in one class a different $k$-subset of $\{1, 2, \ldots, 2k-1\}$, and do the same for the other class. Suppose there is a good choice of colors. Then in each class we have to use at least $k$ different colors (one cannot stab all $k$-subsets with at most $k-1$ numbers). But if in one of the classes we use $k$ different colors, there will be a vertex in the other class that has no other color available. Contradiction.

If you spot any mistakes on this sheet, please drop an email to filip.moric@epfl.ch.