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Due Date: March 09, 2009
Discussions: March 10, 2009

Introduction to Discrete Optimization

Spring 2009

Assignment Sheet 3

Exercise 1

Consider a school district with I neighborhoods, J schools and G grades at each school. Each school j has a capacity of C_{jg} for grade g. In each neighborhood i, the student population of grade g is S_{ig}. Finally the distance of school j from neighborhood i is d_{ij}. Formulate a linear programming problem whose objective is to assign all students to schools, while minimizing the total distance traveled by all students. (You may ignore the fact that numbers of students must be integer.)

Exercise 2

Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let $A = \{x_1, \ldots, x_5\}$. Find two disjoint subsets $A_1,\ A_2 \subseteq A$ such that

$$\text{conv}(A_1) \cap \text{conv}(A_2) \neq \emptyset.$$

Hint: Recall the proof of Radon's lemma

Exercise 3

Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The vector

$$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 = \begin{pmatrix} 15 \\ 14 \\ 25 \end{pmatrix}$$

is a conic combination of the x_i.

Write v as a conic combination using only three vectors of the x_i.

Hint: Recall the proof of Carathéodory's theorem

Exercise 4

Show that a basic solution can be associated to two different bases, i.e. give an example of a solution x^* to a linear program $\min\{c^T x \ : \ Ax = b, x \geq 0\}$ such that there are two bases A_B and $A_{B'}$ with $A_B x^*_B = b$, $A_{B'} x^*_{B'} = b$ and $x^*(i) = 0 \ \forall i \in \{j = 1, \ldots, n \ : \ j \notin B \cup B'\}$.

Exercise 5

Recall the *naive* algorithm given in the lecture to solve a linear program by generating all basic solutions. Consider linear programs of the form

$$\min\{c^T x \;:\; Ax = b,\; x \geq 0\},$$

where $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$ and $c \in \mathbb{Q}^n$.

Assume that you have a computer that for every subset $J \subseteq \{1,\dots,n\}$ can check whether A_J is a basis, compute $x^* = A_J^{-1}b$, check whether $x^* \geq 0$ and compute $c^T x^*$ in 1 msec.

If $n = 2\cdot m$, what is the largest m such that this computer can calculate an optimal solution of the linear program using the naive algorithm in

1. one minute
2. one day
3. one year (365 days)