Exercise 1
A manager of an oil refinery has 8 million barrels of crude oil A and 5 million barrels of crude oil B allocated for production during the coming month.

These resources can be used to make either gasoline, which sells for 38$ per barrel, or home heating oil, which sells for 33$ per barrel. There are three production processes with the following characteristics:

<table>
<thead>
<tr>
<th></th>
<th>Process 1</th>
<th>Process 2</th>
<th>Process 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input crude A</td>
<td>3</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Input crude B</td>
<td>5</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Output gasoline</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Output heating oil</td>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Cost</td>
<td>51$</td>
<td>11$</td>
<td>41$</td>
</tr>
</tbody>
</table>

All quantities are in barrels. For example, with the first process, 3 barrels of crude A and 5 barrels of crude B are used to produce 4 barrels of gasoline and 3 barrels of heating oil at a cost of 51 $.

Formulate a linear programming problem that would help the manager maximize net revenue over the next month.

Exercise 2
Solve the following linear program using the simplex method:

$$\text{max } \frac{3}{4}x + y$$
subject to
$$x + 2y \leq 10$$
$$x + y \leq 7$$
$$y \leq 4$$
$$x, y \geq 0$$

Exercise 3
Recall Lemma 3.5. from the lecture: *If the reduced cost vector of a basis B is non-negative and if $x^*_B \geq 0$, then B is an optimal basis.*

Show that the reverse is not true by giving a linear program in equation standard form and an optimal basis with optimal feasible associated basic solution $x^*$ with a negative entry in the reduced cost vector.

Exercise 4
Recall the tableau
from the lecture where the simplex method with the given pivoting rule cycled.

Use the lexicographic pivoting rule to solve the simplex tableau. It is defined as follows:

1. If $\bar{c} \geq 0$, then **output optimal basis** $B$

2. Otherwise, let $j$ be an index with $\bar{c}(j) < 0$ and let $u = A_B^{-1}A^j$ be the $j$-th column of the system matrix of the tableau. If $u \leq 0$, then **output** $-\infty$, the linear program is unbounded.

3. Otherwise compute the vectors $[A_B^{-1}b(i) | (A_B^{-1}A)_i] / u(i)$ for all $i = 1, \ldots, m$ with $u(i) > 0$. Let $i^*$ be an index for which this vector is lexicographically smallest.

   Note that $[(A_B^{-1}A)_i | A_B^{-1}b(i)]$ is the $i$th row in the tableau. For two vectors $u, v \in \mathbb{R}^n$, the vector $u$ is lexicographically smaller than $v$ ($u \leq_{\text{lex}} v$), if $u = v$ or if the first nonzero component of $u - v$ is strictly negative.

   The index $B_i^*$ leaves the basis and $j$ enters the basis, i.e., the new basis is $B' = B \setminus \{B_i^*\} \cup \{j\}$.

4. Update the tableau to

\[
\begin{array}{c|cccccc}
 x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-\frac{2}{3} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 & 3 \\
\frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 & 0 \\
\frac{1}{7} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]

\[
\frac{c^T - c^T_{B'}A_B^{-1}A}{A_B^{-1}A} \quad -\frac{c^T_{B'}x_{B'}^*}{x_{B'}^*}
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