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Discussions: May 12, 2009

Introduction to Discrete Optimization

Spring 2009

Assignment Sheet 11

Exercise 1

Let $D = (V, A)$ be a directed graph with capacities $u\colon A \to \mathbb{Q}_{\geq 0}$, costs $c\colon A \to \mathbb{Q}$ and external flow $b\colon V \to \mathbb{Q}$.

1. Show how to transform the minimal cost network flow problem for D, u c and b to a minimal cost network flow problem on a graph D' with functions u', c' and b' such that D' does not contain a pair of reverse arcs.

   Explain how to transform an optimal solution for the MCNFP on D' to an optimal solution for the MCNFP on D.

2. Show that there is no feasible flow in D with capacities u and external flow b unless $\sum_{v \in V} b(v) = 0$ holds.

3. Why can we assume that the network has a path from i to j for all $i \neq j \in V$ which is incapacitated?

Exercise 2

Consider a hockey tournament with n teams. The playing schedule is given as a list with m entries, each of the form (u, v, k), meaning that team u and team v play k times against each other. No match can end in a tie, i.e. each game has a winner and a looser. Let x_i be the number of wins of team i.

Given a vector $(x_1, \ldots, x_n)$, we want to know if it reflects a possible tournament outcome. Show how to model this as a network flow problem. Introduce a digraph D, capacities u and an external flow b such that $(x_1, \ldots, x_n)$ is a possible tournament outcome if and only if there is a feasible flow in D subject to u and b.

Exercise 3

Let $D = (V, A)$ be a directed graph with capacities $u\colon A \to \mathbb{Q}_{\geq 0}$ and external flow $b\colon V \to \mathbb{Q}$.

Explain how to find a feasible flow in D subject to u an b efficiently or assert that no feasible flow exists.

Hint: Use a maximum $s-t$-flow algorithm on an auxiliary network $D' = (V', A')$.

Exercise 4

A *matching* in an undirected graph $G = (V, E)$ is a subset $M \subseteq E$ of the edges such that no two edges in M share a common node of V.

Given a cost function $c\colon E \to \mathbb{R}$, the *matching-problem* is to find a matching that maximizes $c(M) = \sum_{e \in M} c(e)$.

A graph is *bipartite*, if there is a partition V_1, V_2 of V, i.e. we have $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$, such that there are no edges between nodes of V_1 and no edges between nodes of V_2.

Show how to formulate the matching-problem for bipartite graphs as a *min-cost-circulation* problem.