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Introduction to Discrete Optimization

Spring 2009

Assignment Sheet 1

Exercise 1 (Linear programming)
A company produces and sells two different products. Our goal is to determine the number of units of each product they should produce during one month, assuming that there is an unlimited demand for the products, but there are some constraints on production capacity and budget.

There are 20000 hours of machine time in the month. Producing one unit takes 3 hours of machine time for the first product and 4 hours for the second product. Material and other costs for producing one unit of the first product amount to 3CHF, while producing one unit of the second product costs 2CHF. The products are sold for 6CHF and 5CHF per unit, respectively. The available budget for production is 4000CHF initially. 25% of the income from selling the first product can be used immediately as additional budget for production, and so can 28% of the income from selling the second product.

1. Formulate a linear program to maximize the profit subject to the described constraints.
2. Solve the linear program graphically by drawing its set of feasible solutions and determining an optimal solution from the drawing.
3. Suppose the company could modernize their production line to get an additional 2000 machine hours for the cost of 400CHF. Would this investment pay off?

Solution

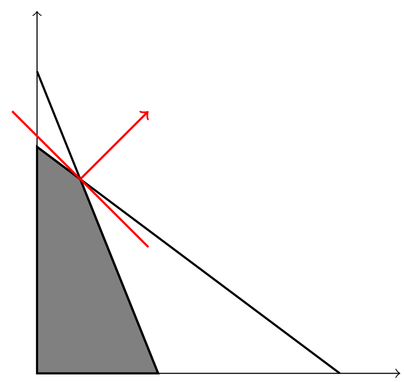
1. Let x be the number of units of the first product and let y be the number of units of the second product.

$$\begin{array}{rl} \max & (6-3)x+(5-2)y \\ \text{subject to} & 3x+4y \leq 20000 \\ & 3x+2y \leq 4000+0.25\cdot 6x+0.28\cdot 5y \\ & x,y \geq 0 \end{array}$$

This can be simplified to:

$$\begin{array}{rl} \max & 3x+3y \\ \text{subject to} & 3x+4y \leq 20000 \\ & 1.5x+0.6y \leq 4000 \\ & x,y \geq 0 \end{array}$$

2. In the following picture, you can see the feasible region and the objective function direction in red.

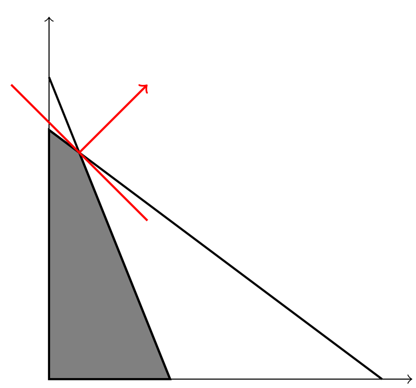


The optimal solution is at the intersection of two of the constraints set to equality, so it is the solution to the system of linear equations:

$$\begin{aligned} 3x + 4y &= 20000 \\ 1.5x + 0.6y &= 4000 \end{aligned}$$

The unique solution is $1000 \cdot (20/21, 30/7)^T$. Its objective function value is $110000/7 \approx 15714$.

3. With an additional 2000 machine hours, the feasible region looks similar – only one constraint has moved, but not by much:



The optimal solution is now the solution of the linear equations:

$$\begin{aligned} 3x + 4y &= 22000 \\ 1.5x + 0.6y &= 4000 \end{aligned}$$

The unique solution is $1000 \cdot (2/3, 5)^T$. Its objective function value is 17000, so even after subtracting the investment of 400CHF the remaining profit is 16600CHF and thus an improvement over the previous optimum.

Exercise 2 (Zimpl)
Recall the image decomposition problem for OLEDs from the lecture and its formulation as a linear program :

$$\begin{aligned} \min\ & \sum_{i=1}^{n} u_i^{(1)} + \sum_{i=1}^{n-1} u_i^{(2)} && && (1) \\ \text{s.t.}\ & f_{ij}^{(1)} + f_{i-1,j}^{(2)} + f_{ij}^{(2)} = r_{ij} && \text{for all } i,j \\ & f_{ij}^{(\alpha)} \le u_i^{(\alpha)} && \text{for all } i,j,\alpha \\ & f_{ij}^{(\alpha)} \ge 0 && \text{for all } i,j,\alpha \end{aligned}$$

Apply thich technique to find an optimal decomposition of the EPFL logo[1] using Zimpl and LP solver libraries:

1. Familiarize yourself with the Zimpl modelling language[2]:

2. Model the linear program 1 to decompose the EPFL logo with Zimpl. You can find an incomplete model containing the encoding of the grayscale values of the logo here[3].

3. Solve the linear program using an LP-solver like QSopt[4], lp_solve[5] or SoPlex[6].

Solution
The linear program (1) can be modelled with zimpl as follows (The zimpl file can be downloaded here[7]):

```
# The image to decompose
set rows := { 1 to 34};
set columns := { 1 to 120};
param r[rows * columns ] := <1,1> 87, <1,2> 87, ... , <34,120> 87;

# The variables of the linear program
var u1[rows] >= -infinity <= infinity;
var u2[rows] >= -infinity <= infinity;
var f1[rows * columns] >= 0 <= infinity;
var f2[rows * columns] >= 0 <= infinity;
```

[1] http://www.epfl.ch/images/EPFL-logo.jpg
[2] http://www.zib.de/koch/zimpl/download/zimpl.pdf
[3] http://disopt.epfl.ch/webdav/site/disopt/users/190205/public/logo_dec.zmpl
[4] http://www2.isye.gatech.edu/ wcook/qsopt/
[5] http://lpsolve.sourceforge.net/5.5/
[6] http://soplex.zib.de/
[7] http://disopt.epfl.ch/webdav/site/disopt/users/190205/public/assignments/logo_dec_sol.zmpl

```
# The objective function
minimize cost: (sum <i> in rows : u1[i]) +  (sum <i> in rows : u2[i]);

# Decomposition constraints
subto dec1: forall<i,j> in rows * columns with i>1 do
    f1[i,j]+f2[i-1,j]+f2[i,j]==r[i,j];
# Special constraint for the first row
subto dec2: forall<j> in columns  do
    f1[1,j]+f2[1,j]==r[1,j];

# bounds for the u-variables
subto bound1: forall<i,j> in rows * columns do
    f1[i,j]<=u1[i];
subto bound2: forall<i,j> in rows * columns do
    f2[i,j]<=u2[i];
```

We can convert the zimpl file to an mps file by typing:

```
zimpl -t mps logo_dec_sol.zmpl
```

This generates the file *logo_dec_sol.mps*, which can be read by lp solver libraries such as *lp_solve*:

```
lp_solve -mps logo_dec_sol.mps
```

This gives an optimal solution to the linear program. The value of the objective function is 4649.5. Thus we have found a decomposition of the image that needs time 4649.5 for display. The traditional approach takes time 8670.

Exercise 3 (LP standard forms)
Let (2) be a linear program in inequality standard form, i.e.

$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\} \tag{2}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Prove that there is an equivalent linear program (3) of the form

$$\max\{\tilde{c}^T x \mid \tilde{A}x = \tilde{b}, x \geq 0, x \in \mathbb{R}^{\tilde{n}}\} \tag{3}$$

where $\tilde{A} \in \mathbb{R}^{\tilde{m}\times\tilde{n}}$, $\tilde{b} \in \mathbb{R}^{\tilde{m}}$, and $\tilde{c} \in \mathbb{R}^{\tilde{n}}$ are such that every feasible point of (2) corresponds to a feasible point of (3) with the same objective function value and vice versa.

Linear programs of the form in (3) are said to be in *equality standard form*.

Solution
The transformation requires two steps:

1. Replace every variable x_j with two non-negative variables x_j^+ and x_j^-, and replace every occurence of x_j with $(x_j^+ - x_j^-)$.

2. Replace every constraint of the form $a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i$ with a constraint $a_{i1}x_1 + \ldots + a_{in} + s_i = b_i$, where s_i is a new, non-negative *slack* variable.

Combining these two steps, we can write the transformed linear program as

$$\begin{array}{rl} \max & c^T x^+ - c^T x^- \\ \text{subject to} & Ax^+ - Ax^- + s = b \\ & x^+ \geq 0 \\ & x^- \geq 0 \\ & s \geq 0 \end{array}$$

This is the desired form if we set $\tilde{c} = \begin{pmatrix} c & -c & 0 \end{pmatrix}$ and $\tilde{A} = \begin{pmatrix} A & -A & I \end{pmatrix}$, where I is the $m \times m$ identity matrix.

Given a feasible solution x of the original linear program, we can find a feasible solution $\tilde{x} = \begin{pmatrix} x^+ & x^- & s \end{pmatrix}^T$ of the reformulated program by setting x^+ to the positive part of x, x^- to the negative part of x, and $s = b - Ax$. It is easy to check that $\tilde{x}$ is feasible and that $\tilde{c}^T \tilde{x} = c^T x$.

Conversely, given a feasible solution $\tilde{x} = \begin{pmatrix} x^+ & x^- & s \end{pmatrix}^T$ of the reformulated program, it is easy to check that $x = x^+ - x^-$ is a feasible solution of the original linear program with the same objective function value.

Exercise 4 (Optimal solutions)
Prove the following statement or give a counterexample: The set of optimal solutions of a linear program is always finite.

Solution
This statement is clearly false. The linear program

$$\max\{x + y \mid x + y \leq 0\}$$

has infinitely many optimal solutions. More generally, if a linear program is bounded, the optimal solutions form a face of the polyhedron of feasible solutions. If this face is at least 1-dimensional, there are infinitely many optimal solutions.