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Introduction to Discrete Optimization

Spring 2009

Solutions 3

Exercise 1

Consider a school district with I neighborhoods, J schools and G grades at each school. Each school j has a capacity of C_{jg} for grade g. In each neighborhood i, the student population of grade g is S_{ig}. Finally the distance of school j from neighborhood i is d_{ij}. Formulate a linear programming problem whose objective is to assign all students to schools, while minimizing the total distance traveled by all students. (You may ignore the fact that numbers of students must be integer.)

Solution

Let x_{ijg} be the amount of students from neighborhood i of grade g travelling to school j. Then for an assignment of students to schools, the total distance travelled by all students is given as

$$\sum_{i \in I} \sum_{j \in J} \sum_{g \in G} d_{ij} x_{ijg}.$$

For a feasible assignment, every student of every neighborhood and grade must be assigned to a school, this gives the constraint

$$\sum_{j \in J} x_{ijg} = S_{ig} \quad \forall i \in I, g \in G.$$

The number of students each school can take of the respective grades is bounded by C_{jg}, thus

$$\sum_{i \in I} x_{ijg} \leq C_{jg} \quad \forall j \in J, g \in G$$

must hold. Finally there can be no negative numbers of assignments: $x \geq 0$. This gives the following linear program:

$$\begin{array}{rll} \min & \sum_{i \in I} \sum_{j \in J} \sum_{g \in G} d_{ij} x_{ijg} & \\ \text{subject to} & \sum_{j \in J} x_{ijg} = S_{ig} & \forall i \in I, g \in G \\ & \sum_{i \in I} x_{ijg} \leq C_{jg} & \forall j \in J, g \in G \\ & x \geq 0 & \end{array}$$

Exercise 2

Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let $A = \{x_1, \ldots, x_5\}$. Find two disjoint subsets $A_1,\ A_2 \subseteq A$ such that

$$\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \neq \emptyset.$$

Hint: Recall the proof of Radon's lemma

Solution
Since we have 5 points in $\mathbb{R}^3$, Radon's lemma states that the subsets A_1 and A_2 exist. To compute them, we review the proof of Radon's lemma.

We construct the set

$$A' = \left\{ \begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \begin{pmatrix} x_2 \\ 1 \end{pmatrix}, \begin{pmatrix} x_3 \\ 1 \end{pmatrix}, \begin{pmatrix} x_4 \\ 1 \end{pmatrix}, \begin{pmatrix} x_5 \\ 1 \end{pmatrix} \right\}$$

The vectors of A' are linearly dependent and we can compute a nontrivial linear combination of the all zero vector.

$$0 = \sum_{i=1}^{5} \lambda_i \begin{pmatrix} x_i \\ 1 \end{pmatrix}.$$

As shown in the proof, if the define sets $P := \{i : \lambda_i \geq 0\}$ and $N := \{i : \lambda_i < 0\}$, then the sets $A_1 := \{x_i : i \in P\}$ and $A_2 := \{x_i : i \in N\}$ have the desired property.

To compute a nontrivial linear combination of the all zero vector using points from A', we solve the following linear program:

$$\begin{aligned} 3\lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 &= 0 \\ \lambda_1 + 2\lambda_2 + 4\lambda_4 + \lambda_5 &= 0 \\ 2\lambda_1 + 5\lambda_2 + \lambda_3 + 3\lambda_4 + \lambda_5 &= 0 \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 &= 0 \end{aligned}$$

Using standard methods, e.g. gaussian elimination, one can compute the solution set as:

$$S = \left\{ (a, 0, -\frac{3}{2}a, -\frac{1}{2}a, a) \ : \ a \in \mathbb{R} \right\}.$$

We take the solution $(1, 0, -\frac{3}{2}, -\frac{1}{2}, 1) \in S$ which gives the nontrivial linear combination

$$0 = \begin{pmatrix} x_1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} x_2 \\ 1 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} x_3 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} x_4 \\ 1 \end{pmatrix} + \begin{pmatrix} x_5 \\ 1 \end{pmatrix}.$$

As shown in the proof of Radon's lemma, the sets $A_1 := \{x_1, x_2, x_5\}$ and $A_2 := \{x_3, x_4\}$ have the required property, i.e $\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \neq \emptyset$.

As a certificate the proof gives that

$$v = \frac{1}{2}x_1 + \frac{1}{2}x_5 = \frac{3}{4}x_3 + \frac{1}{4}x_4 = \begin{pmatrix} 2 \\ 1 \\ 1.5 \end{pmatrix}$$

is contained in both $\operatorname{conv} A_1$ and $\operatorname{conv} A_2$.

Exercise 3
Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The vector

$$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 = \begin{pmatrix} 15 \\ 14 \\ 25 \end{pmatrix}$$

is a conic combination of the x_i.

Write v as a conic combination using only three vectors of the x_i.

Hint: Recall the proof of Carathéodory's theorem

Solution
Let $X = \{x_1, \ldots, x_n\}$. Observe that $v \in \text{cone}(X)$. Since $x_1, \ldots, x_5 \in \mathbb{R}^3$, Carathéodory's theorem states that we can write v as a conic combination using at most three vectors of X.

How to compute this conic combination? Recall the proof of Carathéodory's theorem. The number of vectors in the conic combination $v = \sum_{i=1}^{5} \lambda_i x_i$ can be reduced by one with the following method: Compute a nontrivial linear combination of the all zero vector, i.e. compute $\mu_1, \ldots, \mu_5 \in \mathbb{R}$, not all of them zero such that $\sum_{i=1}^{5} \mu_i x_i = 0$ holds.

Thus $v = \sum_{i=1}^{5} (\lambda_i - \epsilon\mu_i) x_i$ for each $\epsilon > 0$. As described in the proof, one can find an ϵ^* such that $\lambda_i - \epsilon^* \mu_i \geq 0$ for each $i = 1, \ldots, 5$ and $\lambda_i - \epsilon^* \mu_i = 0$ for at least one i. Thus we get a new conic combination of v using one vector less than before.

We now applying the idea to the exercise. We first compute a nontrivial linear combination of the all zero vector by solving the following system of linear equations:

$$\begin{aligned} 3\mu_1 + \mu_2 + 2\mu_3 + 2\mu_4 + \mu_5 &= 0 \\ \mu_1 + 2\mu_2 + 4\mu_4 + \mu_5 &= 0 \\ 2\mu_1 + 5\mu_2 + \mu_3 + 3\mu_4 + \mu_5 &= 0. \end{aligned}$$

Using standard methods, e.g. gaussian elimination, one can compute the solution set as:

$$S = \{(-4a - b, 0, 5a + b, a, b) \;:\; a, b \in \mathbb{R}\}$$

We take the solution $(-5, 0, 6, 1, 1) \in S$ which gives a nontrivial linear combination, i.e. $0 = -5x_1 + 6x_3 + x_4 + x_5$.

What is the maximal ϵ such that

$$v = (1 + 5\epsilon)x_1 + 3x_2 + (2 - 6\epsilon)x_3 + (1 - \epsilon)x_4 + (3 - \epsilon)x_5$$

is a conic combination? Each coefficient has to be nonnegative, thus observe that

$$\epsilon^* = \frac{1}{3}$$

is the maximum. We get the new conic combination

$$v = \frac{8}{3}x_1 + 3x_2 + 0x_3 + \frac{2}{3}x_4 + \frac{8}{3}x_5.$$

Observe that since the coefficient of x_3 is zero, we can remove it from the conic combination.

We need to remove one more vector to get a conic combination using only three vectors. Again we compute a nontrivial linear combination of the all zero vector using the remaining vectors (x_1, x_2, x_4, x_5):

$$\begin{aligned} 3\mu_1 + \mu_2 + 2\mu_4 + \mu_5 &= 0 \\ \mu_1 + 2\mu_2 + 4\mu_4 + \mu_5 &= 0 \\ 2\mu_1 + 5\mu_2 + 3\mu_4 + \mu_5 &= 0. \end{aligned}$$

We compute the solution set which is

$$S' = \{(-a, 0, -a, 5a) \ : \ a \in \mathbb{R}\}$$

We take the solution $(-1, 0, -1, 5) \in S'$ which gives a nontrivial linear combination, i.e. $0 = -x_1 - x_3 + 5x_5$.

What is the maximal ϵ such that

$$v = (\frac{8}{3} + \epsilon)x_1 + 3x_2 + (\frac{2}{3} + \epsilon)x_4 + (\frac{8}{3} - 5\epsilon)x_5$$

is a conic combination? It is given by $\epsilon^* = \frac{8}{15}$.

The new conic combination is

$$\begin{aligned} v &= (\frac{8}{3} + \frac{8}{15})x_1 + 3x_2 + (\frac{2}{3} + \frac{8}{15})x_4 + (\frac{8}{3} - 5\frac{8}{15})x_5 \\ &= \frac{16}{5}x_1 + 3x_2 + \frac{6}{5}x_4 + 0x_5. \end{aligned}$$

Since the coefficient of x_5 is zero, we can remove it and obtain the desired convex combination of v using only three vectors.

Exercise 4

Show that a basic solution can be associated to two different bases, i.e. give an example of a solution x^* to a linear program $\min\{c^T x \ : \ Ax = b, x \geq 0\}$ such that there are two bases A_B and $A_{B'}$ with $A_B x_B^* = b$, $A_{B'} x_{B'}^* = b$ and $x^*(i) = 0 \ \forall i \in \{j = 1, \ldots, n \ : \ j \notin B \cap B'\}$.

Solution

Consider the linear program

$$\begin{aligned} \min \quad & \textstyle\sum_{i=1}^n x_i \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

Where

$$A := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let $B = \{1, 2\}$ and $B' = \{1, 3\}$.

Set $x^* := (1, 0, 0)^T$. Observe that x^* has all desired properties.

Exercise 5

Recall the *naive* algorithm given in the lecture to solve a linear program by generating all basic solutions. Consider linear programs of the form

$$\min\{c^T x \ : \ Ax = b, \ x \geq 0\},$$

where $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$ and $c \in \mathbb{Q}^n$.

Assume that you have a computer that for every subset $J \subseteq \{1, \ldots, n\}$ can check whether A_J is a basis, compute $x^* = A_J^{-1} b$, check whether $x^* \geq 0$ and compute $c^T x^*$ in 1 msec.

If $n = 2 \cdot m$, what is the largest m such that this computer can calculate an optimal solution of the linear program using the naive algorithm in

1. one minute
2. one day
3. one year (365 days)

Solution

The time needed is given by the number of sets we have to test: The naive algorithm considers each subset of $\{1, \ldots, n\}$ of cardinality m. For each such set 1 msec is needed. Since there are $\binom{n}{m} = \binom{2 \cdot m}{m}$ such sets, the naive algorithm needs a running time of $\binom{2 \cdot m}{m}$ msec.

One minute has 60,000 msec. $\binom{18}{9} = 48{,}620$ and $\binom{20}{10} = 184{,}756$. Thus $m = 9$ is the largest m that can be processed in less than one minute.

One day has 86,400,000 msec. $\binom{28}{14} = 40{,}116{,}600$ and $\binom{30}{15} = 155{,}117{,}520$. Thus $m = 14$ is the largest m that can be processed in less than one day.

One year has 31,536,000,000 msec. $\binom{36}{18} = 9{,}075{,}135{,}300$ and $\binom{38}{19} = 35{,}345{,}263{,}800$. Thus $m = 18$ is the largest m that can be processed in less than one year.