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Introduction to Discrete Optimization

Spring 2009

Solutions 4

Exercise 1
A company must deliver d_i units of its product at the end of the ith month. Material produced during a month can be delivered either at the end of the same month or can be stored as inventory and delivered at the end of a subsequent month; however, there is a storage cost of c_1 dollars per month for each unit of product held in inventory. The year begins with zero inventory. If the company produces x_{i-1} units in month $i-1$ and x_i units in month i, it incurs a cost of $c_2 \cdot |x_i - x_{i-1}|$ dollars, reflecting the cost of switching to a new production level. Formulate a linear programming problem whose objective is to minimize the total cost of the production and inventory schedule over a period of twelve months. Assume that inventory left at the end of the year has no value and does not incur any storage costs.

Solution
Let x_i, $i = 1, \dots, 12$, reflect the units of the product produced in month i, and let $x_0 := 0$. Let y_i, $i = 2, \dots, 12$, be the units of the product produced in month $i-1$ and delivered at the end of month i. y_{13} is the amount of product left over after the end of the year. Let $y_1 := 0$.

In each month i the company has to deliver d_i many units. This constitues of the units y_i on stock plus the units x_i produced in month i minus the units y_{i+1} held back for deliver next month. Thus we get the constraint

$$y_i + x_i - y_{i+1} = d_i$$

for each month $i = 1, \dots, 12$.

To make sure that no units of the product are stored for longer than one month, we add the constraint

$$y_i \le d_i.$$

The objective we want to minimize is

$$c_2 \sum_{i=1}^{12} |x_i - x_{i-1}| + c_1 \sum_{i=2}^{12} y_i.$$

This objective function is not linear, since $|.|$ is not linear.

To deal with this, we model the absolute value $|x_i - x_{i-1}|$ by introducing variables z_i and the constraints

$$x_i - x_{i-1} \le z_i$$

and

$$x_{i-1} - x_i \leq z_i.$$

Then the objective function is given as

$$c_2 \sum_{i=1}^{12} z_i + c_1 \sum_{i=2}^{12} y_i.$$

Finally the values of x, y and z have to be nonnegative. The resulting linear program is:

$$\begin{array}{rll} \min & c_2 \sum_{i=1}^{12} z_i + c_1 \sum_{i=2}^{12} y_i & \\ \text{subject to} & y_i + x_i - y_{i+1} = d_i & \forall i = 1, \ldots, 12 \\ & y_i \leq d_i & \forall i = 1, \ldots, 12 \\ & x_i - x_{i-1} \leq z_i & \forall i = 1, \ldots, 12 \\ & x_{i-1} - x_i \leq z_i & \forall i = 1, \ldots, 12 \\ & x, y, z \geq 0 & \end{array}$$

Exercise 2
The vector $x^* = (0, 1, 1, 1)$ is an optimal solution of

$$\begin{array}{rcl} \min (1,1,0,2) \cdot x & & \\ \underbrace{\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix}}_{=A} x & = & \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} \\ x & \geq & \mathbf{0} \end{array}$$

with $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Use the proof of Lemma 3.1 to find another optimal solution x' such that $A_{J'}$ has full column rank with $J' = \{i \mid x'_i > 0\}$.

Solution

Let $J = \{i \mid x^*_i > 0\} = \{2, 3, 4\}$. Observe thast $A_J = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ does not have full column rank.

As seen in the proof, we find a vector $d \in \mathbb{R}^4$ with $Ad = 0$ and $d(j) = 0 \; \forall j \notin J$. Note that every vector d with these properties is contained in

$$\text{kern} \begin{pmatrix} A \\ e_1^T \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 0 \\ -4 \\ -1 \\ 2 \end{pmatrix} \right\}$$

Thus choose $d = (0, -4, -1, 2)$.

Observe that $c^T d = (1,1,0,2) \cdot (0,-4,-1,2)^T = 0$. Choose ε maximal, such that

$$\begin{pmatrix}0\\1\\1\\1\end{pmatrix} + \varepsilon \begin{pmatrix}0\\-4\\-1\\2\end{pmatrix} \geq \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Thus $\varepsilon = 1/4$. Then

$$x' := x + \varepsilon d = \begin{pmatrix}0\\1\\1\\1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0\\-4\\-1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\3/4\\3/2\end{pmatrix}$$

Now $J' = \{3,4\}$ thus

$$A_{J'} = \begin{pmatrix}2 & 3\\0 & 2\\0 & 0\end{pmatrix}$$

which has full column rank.

Exercise 3
Solve the following tableau with the simplex algorithm:

x_1	x_2	x_3	x_4	x_5	
-3	-2	-1	0	0	0
1	2	3	1	0	3
1	-1	2	0	1	2

For each iteration give the simplex tableau, the current basis and the indices leaving/entering the basis.

Solution
The initial basis is $B = \{x_4, x_5\}$. The reduced costs of x_1, x_2 and x_3 are all negative and thus could be chosen to enter the basis. We choose x_1. Both entries of x_1's are positive, but only removing x_5 gives a feasible solution. Thus we get the new tableau

x_1	x_2	x_3	x_4	x_5	
0	-5	5	0	3	6
0	3	1	1	-1	1
1	-1	2	0	1	2

with basis $B = \{x_1, x_4\}$.

Now only x_2 has negative reduced costs. Thus x_2 enters the basis and x_4 leaves the basis. The new tableau is

x_1	x_2	x_3	x_4	x_5	
0	0	$\frac{20}{3}$	$\frac{5}{3}$	$\frac{10}{3}$	$\frac{23}{3}$
0	1	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{7}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{7}{3}$

All reduced costs are nonnegative. We have found the otimal solution $x^* = (\frac{7}{3}, \frac{1}{3}, 0, 0, 0)$ of value $-\frac{23}{3}$.

Exercise 4
Solve the following linear program by using the simplex method

$$\begin{aligned} \min \begin{pmatrix} -4 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}^T x & & \\ \begin{pmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} x &= \begin{pmatrix} 6 \\ 9 \\ 2 \end{pmatrix} \\ x &\geq \mathbf{0} \end{aligned}$$

Start with the basis $B = \{4, 5, 6\}$. For each iteration give the simplex tableau and the indices leaving/entering the basis.

Solution
Since A_B is the identity matrix, A_B^-1 is the identity matrix aswell and the initial simplex tableau is given as follows:

x_1	x_2	x_3	x_4	x_5	x_6	
-4	-1	1	0	0	0	0
1	0	3	1	0	0	6
3	1	3	0	1	0	9
1	1	-1	0	0	1	2

We choose x_1 to enter the basis. Thus x_6 must leave the basis. The new tableau is:

x_1	x_2	x_3	x_4	x_5	x_6	
0	3	-3	0	0	4	8
0	-1	4	1	0	-1	4
0	-2	6	0	1	-3	3
1	1	-1	0	0	1	2

The new basis is $B = \{x_1, x_4, x_5\}$. Next x_3 enters the basis, and x_5 has to leave.

x_1	x_2	x_3	x_4	x_5	x_6	
0	2	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{19}{2}$
0	$\frac{1}{3}$	0	1	$-\frac{2}{3}$	1	2
0	$-\frac{1}{3}$	1	0	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{2}$
1	$\frac{2}{3}$	0	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{5}{2}$

The new basis is $B = \{x_1, x_3, x_4\}$.

There are no more negative reduced costs. The optimum solution is $x^* = (\frac{5}{2}, 0, \frac{1}{2}, 2, 0, 0)$ with $c^T x^* = -\frac{19}{2}$.

Exercise 5
Suppose you are given an oracle algorithm, which for a given polyhedron $P = \{\tilde{x} \in \mathbb{R}^{\tilde{n}} \ : \ \tilde{A}\tilde{x} \leq$

$\tilde{b}\}$ gives you a feasible solution or asserts that there is none. Show that using a single call of this oracle one can obtain an optimum solution for the LP

$$\min\{c^T x \,:\, x \in \mathbb{R}^n;\ Ax = b;\ x \geq \mathbf{0}\},$$

assuming that the LP is feasible and bounded.
Hint: Use duality.

Solution
The LP is feasible and bounded, thus an optimum solution must exist. Strong duality tells us that the dual $\max\{b^T y \mid A^T y \leq c\}$ is feasible and bounded, and for optimal solutions x^* of the primal and y^* of the dual we have $b^T y^* = c^T x^*$.

Thus every point (x^*, y^*) of the polyhedron

$$\begin{aligned} c^T x &= b^T y \\ Ax &= b \\ A^T y &\leq c \\ x &\geq 0 \end{aligned}$$

is optimal. Hence with one oracle call for the polyhedron above we get an optimal solution of the LP.