Exercise 1
A manager of an oil refinery has 8 million barrels of crude oil A and 5 million barrels of crude oil B allocated for production during the coming month.

These resources can be used to make either gasoline, which sells for 38$ per barrel, or home heating oil, which sells for 33$ per barrel. There are three production processes with the following characteristics:

<table>
<thead>
<tr>
<th></th>
<th>Process 1</th>
<th>Process 2</th>
<th>Process 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input crude A</td>
<td>3</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Input crude B</td>
<td>5</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Output gasoline</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Output heating oil</td>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Cost</td>
<td>51$</td>
<td>11$</td>
<td>41$</td>
</tr>
</tbody>
</table>

All quantities are in barrels. For example, with the first process, 3 barrels of crude A and 5 barrels of crude B are used to produce 4 barrels of gasoline and 3 barrels of heating oil at a cost of 51 $.

Formulate a linear programming problem that would help the manager maximize net revenue over the next month.

Solution
We introduce variables $x_1$, $x_2$ and $x_3$ for the three production processes.

Since the amount of crude $A$ is limited by 8 million barrels, we get the constraint

$$3x_1 + x_2 + 5x_3 \leq 8.$$ 

Similarly, we get the constraint

$$5x_1 + x_2 + 3x_3 \leq 5$$

for crude $B$.

The net revenue (in million $) is given as

$$38 \cdot (4x_1 + x_2 + 3x_3) + 33 \cdot (3x_1 + x_2 + 4x_3) - 51x_1 - 11x_2 - 41x_3.$$ 

This can be simplified to

$$200x_1 + 60x_2 + 205x_3.$$
Thus the linear program is given as:

\[
\begin{align*}
\text{max } & \quad 200x_1 + 60x_2 + 205x_3 \\
\text{subject to } & \quad 3x_1 + x_2 + 5x_3 \leq 8 \\
& \quad 5x_1 + x_2 + 3x_3 \leq 5 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

**Exercise 2**
Solve the following linear program using the simplex method:

\[
\begin{align*}
\text{max } & \quad \frac{3}{4}x + y \\
\text{subject to } & \quad x + 2y \leq 10 \\
& \quad x + y \leq 7 \\
& \quad y \leq 4 \\
& \quad x, y \geq 0
\end{align*}
\]

**Solution**
First we have to transform the linear program into equation standard form by introducing slack variables and rewriting it as a minimization problem:

\[
\begin{align*}
\text{min } & \quad -\frac{3}{4}x - y \\
\text{subject to } & \quad x + 2y + s = 10 \\
& \quad x + y + t = 7 \\
& \quad y + u = 4 \\
& \quad x, y, s, t, u \geq 0
\end{align*}
\]

We start with the initial basis \( B = \{ s, t, u \} \) and the initial solution \( x = y = 0, s = 10, t = 7 \) and \( u = 4 \). The corresponding tableau looks as follows

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( s )</th>
<th>( t )</th>
<th>( u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

We choose \( y \) to enter the basis. \( 4 < \frac{10}{7} < 7 \), and thus \( u \) has to leave the basis. We get:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( s )</th>
<th>( t )</th>
<th>( u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
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<td>1</td>
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<td>1</td>
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<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

2
After permuting the rows we obtain:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>s</th>
<th>t</th>
<th>u</th>
</tr>
</thead>
<tbody>
<tr>
<td>-\frac{3}{4}</td>
<td>0 0 0 1 4</td>
<td>0 1 0 0 1 4</td>
<td>1 0 1 0 -2 2</td>
<td>1 0 0 1 -1 3</td>
</tr>
</tbody>
</table>

Now $x$ enters the basis. $2 < 3$ and thus $s$ leaves the basis:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>s</th>
<th>t</th>
<th>u</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 \frac{3}{4} 0 -\frac{1}{2} \frac{11}{7}</td>
<td>0 1 0 0 1 4</td>
<td>1 0 1 0 -2 2</td>
<td>0 0 -1 1 1 1</td>
<td></td>
</tr>
</tbody>
</table>

After permuting we get:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>s</th>
<th>t</th>
<th>u</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 \frac{3}{4} 0 -\frac{1}{2} \frac{11}{7}</td>
<td>0 1 0 0 1 4</td>
<td>1 0 1 0 -2 2</td>
<td>0 0 -1 1 1 1</td>
<td></td>
</tr>
</tbody>
</table>

The reduced cost of $u$ is negative. Thus $u$ has to enter the basis again. $t$ has to leave:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>s</th>
<th>t</th>
<th>u</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 \frac{1}{3} \frac{1}{2} 0 6</td>
<td>1 0 -1 2 0 4</td>
<td>0 1 1 -1 0 3</td>
<td>0 0 -1 1 1 1</td>
<td></td>
</tr>
</tbody>
</table>

Now all reduced costs are nonnegative and we found the optimal solution $(4, 3, 0, 0, 1)$.

This gives the optimal solution $x = 4, y = 3$ for the original LP of objective value 6.

**Exercise 3**
Recall Lemma 3.5. from the lecture: If the reduced cost vector of a basis $B$ is non-negative and if $x_B^* \geq 0$, then $B$ is an optimal basis.

Show that the reverse is not true by giving a linear program in equation standard form and an optimal basis with optimal feasible associated basic solution $x^*$ with a negative entry in the reduced cost vector.

**Solution**
Consider the linear program

\[
\begin{align*}
\text{min } & \quad x_1 - 10x_2 - 20x_3 \\
\text{subject to } & \quad x_1 + x_3 = 1 \\
& \quad x_2 + x_3 = 0 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]
Observe that the second constraint and the nonnegativity of the variables requires $x_2 = x_3 = 0$ for every feasible solution. Thus the first constraint requires $x_1 = 1$ for every feasible solution. Thus there is only one feasible solution to the system, namely $x^* = (1, 0, 0)$, which hence is optimal.

$x^*$ is associated basic solution to $B = \{1, 2\}$.

Here $c = (1, -10, -20)^T$ and $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Thus the reduced cost of $x_3$ is given as $c_3 - c_B^T A_B^{-1} A_3 = -20 - 1 \cdot 1 + 10 \cdot 1 = -11 < 0$

**Exercise 4**
Recall the tableau

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\frac{2}{3}$</td>
<td>20</td>
<td>$-\frac{1}{2}$</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{3}$</td>
<td>$-8$</td>
<td>$-\frac{1}{2}$</td>
<td>9</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$-12$</td>
<td>$-\frac{1}{2}$</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

from the lecture where the simplex method with the given pivoting rule cycled.

Use the lexicographic pivoting rule to solve the simplex tableau. It is defined as follows:

1. If $\bar{c} \geq 0$, then **output optimal basis** $B$.

2. Otherwise, let $j$ be an index with $\bar{c}(j) < 0$ and let $u = A_B^{-1} A^j$ be the $j$-th column of the system matrix of the tableau. If $u \leq 0$, then **output** $-\infty$, the linear program is unbounded.

3. Otherwise compute the vectors $[(A_B^{-1} b)(i)\, (A_B^{-1} A)_i]/u(i)$ for all $i = 1, \ldots, m$ with $u(i) > 0$. Let $i^*$ be an index for which this vector is lexicographically smallest.

Note that $[(A_B^{-1} A)_i\, (A_B^{-1} b)(i)]$ is the $i$th row in the tableau. For two vectors $u, v \in \mathbb{R}^n$, the vector $u$ is lexicographically smaller than $v$ ($u \leq_{\text{lex}} v$), if $u = v$ or if the first nonzero component of $u - v$ is strictly negative.

The index $B_{i^*}$ leaves the basis and $j$ enters the basis, i.e., the new basis is $B' = B \setminus \{B_{i^*}\} \cup \{j\}$.

4. Update the tableau to

\[
\begin{array}{c|c|c}
\hline
A_{B'}^{-1} A & c_T - c_{B'}^T A_{B'}^{-1} A_{B'} & -c_{B'}^T x_{B'}^* \\
\hline
x_{B'}^* & & \\
\hline
\end{array}
\]

**Solution**

The initial basis is $B = \{5, 6, 7\}$. We choose 1 to enter the basis. The first row is the lexicographic smallest row ($-8 \cdot 4 < -12 \cdot 2$) and thus 5 leaves the basis. We get:

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-\frac{4}{3}$</td>
<td>33</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>$-32$</td>
<td>$-4$</td>
<td>36</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>$\frac{5}{2}$</td>
<td>$-15$</td>
<td>$-2$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Now we choose 2 to enter the basis, 6 leaves the basis 

\[
\begin{array}{cccccccc}
\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 & \mathbf{x}_5 & \mathbf{x}_6 & \mathbf{x}_7 \\
0 & 0 & -2 & 18 & 1 & 1 & 0 & 3 \\
1 & 0 & 8 & -84 & -12 & 8 & 0 & 0 \\
0 & 1 & \frac{3}{8} & -\frac{15}{4} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1
\end{array}
\]

Now we choose 3 to enter the basis, according to the lexicographic rule, 2 leaves the basis. We get:

\[
\begin{array}{cccccccc}
\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 & \mathbf{x}_5 & \mathbf{x}_6 & \mathbf{x}_7 \\
0 & \frac{16}{3} & 0 & -2 & -\frac{3}{2} & \frac{7}{3} & 0 & 3 \\
1 & -\frac{64}{3} & 0 & -4 & -\frac{4}{3} & \frac{22}{3} & 0 & 0 \\
0 & \frac{8}{3} & 1 & -10 & -\frac{4}{3} & \frac{22}{3} & 0 & 0 \\
0 & -\frac{4}{3} & 0 & 10 & \frac{4}{3} & -\frac{2}{3} & 1 & 1
\end{array}
\]

Now we choose 4 to enter the basis. 7 must leave the basis and we get

\[
\begin{array}{cccccccc}
\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 & \mathbf{x}_5 & \mathbf{x}_6 & \mathbf{x}_7 \\
0 & \frac{24}{5} & 0 & 0 & -\frac{7}{5} & \frac{11}{5} & \frac{1}{5} & \frac{16}{5} \\
1 & -\frac{112}{5} & 0 & 0 & -\frac{2}{5} & \frac{12}{5} & \frac{2}{5} & \frac{2}{5} \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & -\frac{4}{15} & 0 & 1 & \frac{7}{15} & -\frac{1}{15} & \frac{1}{10} & \frac{1}{10}
\end{array}
\]

Now 5 enters the basis. Thus 4 must leave. We get:

\[
\begin{array}{cccccccc}
\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 & \mathbf{x}_5 & \mathbf{x}_6 & \mathbf{x}_7 \\
0 & 2 & 0 & \frac{21}{2} & 0 & \frac{3}{2} & \frac{3}{4} & \frac{12}{7} \\
1 & -24 & 0 & 6 & 0 & 2 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & -2 & 0 & \frac{13}{2} & 1 & -\frac{1}{2} & \frac{3}{4} & \frac{3}{2}
\end{array}
\]

All reduced costs are nonnegative. We are done. The optimal solution is 

\[ (1, 0, 1, 0, \frac{3}{4}, 0, 0) \]

of value \(-\frac{17}{4}\).