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Discussions: March 24, 2009

Introduction to Discrete Optimization

Spring 2009

Solutions 5

Exercise 1
A manager of an oil refinery has 8 million barrels of crude oil A and 5 million barrels of crude oil B allocated for production during the coming month.

These resources can be used to make either gasoline, which sells for 38$ per barrel, or home heating oil, which sells for 33$ per barrel. There are three production processes with the following characteristics:

	Process 1	Process 2	Process 3
Input crude A	3	1	5
Input crude B	5	1	3
Output gasoline	4	1	3
Output heating oil	3	1	4
Cost	51$	11$	41$

All quantities are in barrels. For example, with the first process, 3 barrels of crude A and 5 barrels of crude B are used to produce 4 barrels of gasoline and 3 barrels of heating oil at a cost of 51 $.

Formulate a linear programming problem that would help the manager maximize net revenue over the next month.

Solution
We introduce variables x_1, x_2 and x_3 for the three production processes.

Since the amount of crude A is limited by 8 million barrels, we get the constraint

$$3x_1 + x_2 + 5x_3 \leq 8.$$

Similarly, we get the constraint

$$5x_1 + x_2 + 3x_3 \leq 5$$

for crude B.

The net revenue (in millon $) is given as

$$38 \cdot (4x_1 + x_2 + 3x_3) + 33 \cdot (3x_1 + x_2 + 4x_3) - 51x_1 - 11x_2 - 41x_3.$$

This can be simplified to

$$200x_1 + 60x_2 + 205x_3.$$

Thus the linear program is given as:

$$\begin{aligned} \max \quad & 200x_1 + 60x_2 + 205x_3 \\ \text{subject to} \quad & 3x_1 + x_2 + 5x_3 \leq 8 \\ & 5x_1 + x_2 + 3x_3 \leq 5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Exercise 2
Solve the following linear program using the simplex method:

$$\begin{aligned} \max \quad & \tfrac{3}{4}x + y \\ \text{subject to} \quad & x + 2y \leq 10 \\ & x + y \leq 7 \\ & y \leq 4 \\ & x, y \geq 0 \end{aligned}$$

Solution
First we have to transform the linear program into equation standard form by introducing slack variables and rewriting it as a minimization problem:

$$\begin{aligned} \min \quad & -\tfrac{3}{4}x - y \\ \text{subject to} \quad & x + 2y + s = 10 \\ & x + y + t = 7 \\ & y + u = 4 \\ & x, y, s, t, u \geq 0 \end{aligned}$$

We start with the initial basis $B = \{s, t, u\}$ and the initial solution $x = y = 0$, $s = 10$, $t = 7$ and $u = 4$. The corresponding tableau looks as follows

x	y	s	t	u	
$-\frac{3}{4}$	-1	0	0	0	0
1	2	1	0	0	10
1	1	0	1	0	7
0	1	0	0	1	4

We choose y to enter the basis. $4 < \frac{10}{2} < 7$, and thus u has to leave the basis. We get:

x	y	s	t	u	
$-\frac{3}{4}$	0	0	0	1	4
1	0	1	0	-2	2
1	0	0	1	-1	3
0	1	0	0	1	4

After permuting the rows we obtain:

x	y	s	t	u	
$-\frac{3}{4}$	0	0	0	1	4
0	1	0	0	1	4
1	0	1	0	-2	2
1	0	0	1	-1	3

Now x enters the basis. $2 < 3$ and thus s leaves the basis:

x	y	s	t	u	
0	0	$\frac{3}{4}$	0	$-\frac{1}{2}$	$\frac{11}{2}$
0	1	0	0	1	4
1	0	1	0	-2	2
0	0	-1	1	1	1

After permuting we get:

x	y	s	t	u	
0	0	$\frac{3}{4}$	0	$-\frac{1}{2}$	$\frac{11}{2}$
1	0	1	0	-2	2
0	1	0	0	1	4
0	0	-1	1	1	1

The reduced cost of u is negative. Thus u has to enter the basis again. t has to leave:

x	y	s	t	u	
0	0	$\frac{1}{4}$	$\frac{1}{2}$	0	6
1	0	-1	2	0	4
0	1	1	-1	0	3
0	0	-1	1	1	1

Now all reduced costs are nonnegative and we found the optimal solution $(4, 3, 0, 0, 1)$.

This gives the optimal solution $x = 4, y = 3$ for the original LP of objective value 6.

Exercise 3

Recall Lemma 3.5. from the lecture: *If the reduced cost vector of a basis B is non-negative and if $x^*_B \geq 0$, then B is an optimal basis.*

Show that the reverse is not true by giving a linear program in equation standard form and an optimal basis with optimal feasible associated basic solution x^* with a negative entry in the reduced cost vector.

Solution

Consider the linear program

$$\begin{aligned} \min \quad & x_1 - 10x_2 - 20x_3 \\ \text{subject to} \quad & x_1 + x_3 = 1 \\ & x_2 + x_3 = 0 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Observe that the second constraint and the nonnegativity of the variables requires $x_2 = x_3 = 0$ for every feasible solution. Thus the first constraint requires $x_1 = 1$ for every feasible solution. Thus there is only one feasible solution to the system, namely $x^* = (1,0,0)$, which hence is optimal.

x^* is associated basic solution to $B = \{1,2\}$.

Here $c = (1,-10,-20)^T$ and $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Thus the reduced cost of x_3 is given as $c_3 - c_B^T A_B^{-1} A_3 = -20 - 1 \cdot 1 + 10 \cdot 1 = -11 < 0$

Exercise 4

Recall the tableau

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
$-\frac{3}{4}$	20	$-\frac{1}{2}$	6	0	0	0	3
$\frac{1}{4}$	-8	-1	9	1	0	0	0
$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0	1	0	0
0	0	1	0	0	0	1	1

from the lecture where the simplex method with the given pivoting rule cycled.

Use the lexicographic pivoting rule to solve the simplex tableau. It is defined as follows:

1. If $\overline{c} \geq 0$, then **output optimal basis** B
2. Otherwise, let j be an index with $\overline{c}(j) < 0$ and let $u = A_B^{-1} A^j$ be the j-th column of the system matrix of the tableau. If $u \leq 0$, then **output** $-\infty$, the linear program is unbounded.
3. Otherwise compute the vectors $[(A_B^{-1} b)(i) | (A_B^{-1} A)_i] / u(i)$ for all $i = 1, \ldots, m$ with $u(i) > 0$. Let i^* be an index for which this vector is lexicographically smallest.

   Note that $[(A_B^{-1} A)_i | (A_B^{-1} b)(i)]$ is the ith row in the tableau. For two vectors $u, v \in \mathbb{R}^n$, the vector u is lexicographically smaller than v ($u \leq_{lex} v$), if $u = v$ or if the first nonzero component of $u - v$ is strictly negative.

   The index B_{i^*} leaves the basis and j enters the basis, i.e., the new basis is $B' = B \setminus \{B_{i^*}\} \cup \{j\}$.
4. Update the tableau to

$$\begin{array}{c|c} c^T - c_{B'}^T A_{B'}^{-1} A & -c_{B'}^T x_{B'}^* \\ \hline A_{B'}^{-1} A & x_{B'}^*. \end{array}$$

Solution

The initial basis is $B = \{5,6,7\}$. We choose 1 to enter the basis. The first row is the lexicographic smallest row $(-8 \cdot 4 < -12 \cdot 2)$ and thus 5 leaves the basis. We get:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
0	-4	$-\frac{7}{2}$	33	3	0	0	3
1	-32	-4	36	4	0	0	0
0	4	$\frac{3}{2}$	-15	-2	1	0	0
0	0	1	0	0	0	1	1

Now we choose 2 to enter the basis, 6 leaves the basis

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
0	0	-2	18	1	1	0	3
1	0	8	-84	-12	8	0	0
0	1	$\frac{3}{8}$	$-\frac{15}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	0	0
0	0	1	0	0	0	1	1

Now we choose 3 to enter the basis, according to the lexicographic rule, 2 leaves the basis. We get:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
0	$\frac{16}{3}$	0	-2	$-\frac{5}{3}$	$\frac{7}{3}$	0	3
1	$-\frac{64}{3}$	0	-4	$-\frac{4}{3}$	$\frac{8}{3}$	0	0
0	$\frac{8}{3}$	1	-10	$-\frac{4}{3}$	$\frac{2}{3}$	0	0
0	$-\frac{8}{3}$	0	10	$\frac{4}{3}$	$-\frac{2}{3}$	1	1

Now we choose 4 to enter the basis. 7 must leave the basis and we get

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
0	$\frac{24}{5}$	0	0	$-\frac{7}{5}$	$\frac{11}{5}$	$\frac{1}{5}$	$\frac{16}{5}$
1	$-\frac{112}{5}$	0	0	$-\frac{4}{5}$	$\frac{12}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
0	0	1	0	0	0	1	1
0	$-\frac{4}{15}$	0	1	$\frac{2}{15}$	$-\frac{1}{15}$	$\frac{1}{10}$	$\frac{1}{10}$

Now 5 to enters the basis. Thus 4 must leave. We get:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
0	2	0	$\frac{21}{2}$	0	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{17}{4}$
1	-24	0	6	0	2	1	1
0	0	1	0	0	0	1	1
0	-2	0	$\frac{15}{2}$	1	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$

All reduced costs are nonnegative. We are done. The optimal solution is

$$(1,0,1,0,\frac{3}{4},0,0)$$

of value $-\frac{17}{4}$.