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Discussions: March 31, 2009

Introduction to Discrete Optimization

Spring 2009

Solutions 6

Exercise 1

A company produces two kinds of products. A product of the first type requires $\frac{1}{4}$ hours of assembly labor, $\frac{1}{8}$ hours of testing, and 1.2 CHF worth of raw materials. A product of the second type requires $\frac{1}{3}$ hours of assembly, $\frac{1}{3}$ hours of testing and 0.9 CHF worth of raw materials. Given the current personnel of the company, there can be at most 90 hours of assembly labor and 80 hours of testing, each day. Products of the first and second type have a market value of 9 CHF and 8 CHF respectively.

1. Formulate a linear programming problem that can be used to maximize the daily profit of the company.

2. Consider the following two modifications to the original problem:

   a) Suppose that up to 50 hours of overtime assembly labor can be scheduled, at a cost of 7 CHF per hour.

   b) Suppose that the raw material supplier provides a 10% discount if the daily bill is above 300 CHF.

   Which of the above two elements can be easily incorporate into the linear programming formulation and how? If one or both are not easy to incorporoeate, indicate how you might nevertheless solve the problem.

Solution

1. Let x and y denote the number of products of the first and second type produced respectively. The hours needed for assembly work is thus given as $\frac{1}{4}x + \frac{1}{8}y$. Similarly, the hours needed for testing is given as $\frac{1}{3}x + \frac{1}{3}y$. One unit of product one sells for 9 CHF, but requires raw materials of worth 1.2 CHF. Thus the profit for selling one unit of product 1 is 7.8 CHF. Similarly, the profit for selling one unit of product 2 is 7.1 CHF.

   We get the following linear program:

$$\begin{aligned} \max \quad & 7.8x + 7.1y \\ \text{subject to} \quad & \tfrac{1}{4}x + \tfrac{1}{3}y \le 90 \\ & \tfrac{1}{8}x + \tfrac{1}{3}y \le 80 \\ & x, y \ge 0 \end{aligned} \tag{1}$$

2. a) By introducing another variable z giving the number of overtime in assembly we can model the modified problem as follows:

$$\begin{aligned} \max \quad & 7.8x + 7.1y - 7z \\ \text{subject to} \quad & \tfrac{1}{4}x + \tfrac{1}{3}y - z \le 90 \\ & \tfrac{1}{8}x + \tfrac{1}{3}y \le 80 \\ & z \le 50 \\ & x, y, z \ge 0 \end{aligned}$$

b) This change cannot be incorporated into a single linear programming problem. Instead we solve the problem using two linear programs. First add the constraint $1.2x + 0.9y \le 300$ to the linear program (1). An optimal solution to this modified LP gives the best profit under the assumption that the discount is not used. For the second LP to (1), we introduce the constraint $1.2x + 0.9y \ge 300$. This ensures that the discount is used. Next, we modify the objective function to reflect the discount: $(9 - 0.9 \cdot 1.2)x + (8 - 0.9 \cdot 0.9)y$. The optimal solution to the problem is the best solution of both LPs.

Exercise 2

Solve the following linear program using the simplex method:

$$\begin{aligned} \min \quad & -15x_1 - 60x_2 - 4x_3 - 20x_4 \\ \text{subject to} \quad & 20x_1 + 20x_2 + 10x_3 + 40x_4 \le 21 \\ & 10x_1 + 30x_2 + 20x_3 \le 6 \\ & 20x_1 + 40x_2 + 30x_3 + 10x_4 \le 14 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

Hint: Introduce slack variables to transform the LP into equality standard form

Solution

First we have to transform the linear program into equation standard form by introducing slack variables and rewriting it as a minimization problem:

$$\begin{aligned} \min \quad & -15x_1 - 60x_2 - 4x_3 - 20x_4 \\ \text{subject to} \quad & 20x_1 + 20x_2 + 10x_3 + 40x_4 + y_1 = 21 \\ & 10x_1 + 30x_2 + 20x_3 + y_2 = 6 \\ & 20x_1 + 40x_2 + 30x_3 + 10x_4 + y_3 = 14 \\ & x_1, x_2, x_3, x_4, y_1, y_2, y_3 \ge 0 \end{aligned}$$

We start with the initial basis $B = \{y_1, y_2, y_3\}$ and the initial solution $x_1 = x_2 = x_3 = x_4 = 0$, $y_1 = 21$, $y_2 = 6$ and $y_3 = 14$. The corresponding tableau looks as follows

x_1	x_2	x_3	x_4	y_1	y_2	y_3	
-15	-60	-4	-20	0	0	0	0
20	20	10	40	1	0	0	21
10	30	20	0	0	1	0	6
20	40	30	10	0	0	1	14

We choose x_2 to enter the basis. Thus y_2 has to leave the basis and we get:

x_1	x_2	x_3	x_4	y_1	y_2	y_3	
5	0	36	-20	0	2	0	12
$\frac{40}{3}$	0	$-\frac{10}{3}$	40	1	$-\frac{2}{3}$	0	17
$\frac{1}{3}$	1	$\frac{2}{3}$	0	0	$\frac{1}{30}$	0	$\frac{1}{5}$
$\frac{20}{3}$	0	$-\frac{20}{3}$	10	0	-40	1	6

After permuting the rows we obtain:

x_1	x_2	x_3	x_4	y_1	y_2	y_3	
5	0	36	-20	0	2	0	12
$\frac{1}{3}$	1	$\frac{2}{3}$	0	0	$\frac{1}{30}$	0	$\frac{1}{5}$
$\frac{40}{3}$	0	$-\frac{10}{3}$	40	1	$-\frac{2}{3}$	0	17
$\frac{20}{3}$	0	$-\frac{20}{3}$	10	0	-40	1	6

Now x_4 enters the basis. Thus y_1 has to leave:

x_1	x_2	x_3	x_4	y_1	y_2	y_3	
$\frac{35}{3}$	0	$\frac{103}{3}$	0	$\frac{1}{2}$	$\frac{5}{3}$	0	$\frac{41}{2}$
$\frac{1}{3}$	1	$\frac{2}{3}$	0	0	$\frac{1}{30}$	0	$\frac{1}{5}$
$\frac{1}{3}$	0	$-\frac{1}{12}$	1	$\frac{1}{40}$	$-\frac{1}{60}$	0	$\frac{17}{40}$
$\frac{20}{6}$	0	$-\frac{35}{6}$	0	$-\frac{1}{4}$	$-\frac{239}{6}$	1	$\frac{7}{4}$

All reduced costs are nonnegative and thus we found the optimal solution $(0, \frac{1}{5}, 0, \frac{17}{40}, 0, 0, \frac{7}{4})$ with objective value $\frac{41}{2}$.

This gives the optimal solution $x_1 = 0$, $x_2 = \frac{1}{5}$, $x_3 = 0$, $x_4 = \frac{17}{40}$ for the original LP.

Exercise 3

Given the following linear program

$$\begin{aligned} \max \quad & 3x_1 + 4x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 4 \\ & 2x_1 - x_2 \leq -2 \\ & x_1 - 3x_2 \leq 7 \\ & 4x_1 - 5x_2 \leq 10 \\ & 2x_1 + 6x_2 \leq -12 \\ & x_1, x_2 \leq 0 \end{aligned} \tag{2}$$

compute the value of an optimal solution to (2) following these steps:

1. Formulate the dual linear program.
2. Find an initial solution to the dual LP
3. Solve the dual LP using the simplex method.

4. Obtain the optimal value for the LP (2).

   Hint: Strong duality

Solution

The dual linear program is

$$\begin{aligned} \min \quad & 4y_1 - 2y_2 + 7y_3 + 10y_4 - 12y_5 \\ \text{subject to} \quad & -y_1 + 2y_2 + y_3 + 4y_4 + 2y_5 + y_6 = 3 \\ & y_1 - y_2 - 3y_3 - 5y_4 - 6y_5 + y_7 = 4 \\ & y_1, y_2, y_3, y_4, y_5, y_6, y_7 \geq 0 \end{aligned}$$

This suggests to choose $B = \{6, 7\}$ as an initial basis. The corresponding tableau then looks as follows:

y_1	y_2	y_3	y_4	y_5	y_6	y_7	
4	-2	7	10	-12	0	0	0
-1	2	1	4	2	1	0	3
1	-1	-3	-5	6	0	1	4

We let 5 enter the basis, thus 7 has to leave. We get:

y_1	y_2	y_3	y_4	y_5	y_6	y_7	
6	-4	1	0	0	0	2	8
$-\frac{4}{3}$	$\frac{7}{3}$	2	$\frac{17}{3}$	0	1	$-\frac{1}{3}$	$\frac{5}{3}$
$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{5}{6}$	1	0	$\frac{1}{6}$	$\frac{2}{3}$

We let 2 enter the basis, thus 6 has to leave. We get:

y_1	y_2	y_3	y_4	y_5	y_6	y_7	
$\frac{26}{7}$	0	$\frac{31}{7}$	$\frac{68}{7}$	0	$\frac{12}{7}$	$\frac{10}{7}$	$\frac{76}{7}$
$-\frac{4}{7}$	1	$\frac{6}{7}$	$\frac{17}{7}$	0	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{5}{7}$
$\frac{1}{14}$	0	$-\frac{5}{14}$	$-\frac{3}{7}$	1	$\frac{1}{14}$	$\frac{1}{7}$	$\frac{11}{14}$

This shows that the dual is feasible and bounded. The optimal objective value is $-\frac{76}{7}$. Using strong duality, we get that the optimal value of the primal is $-\frac{76}{7}$ as well.

Exercise 4

Consider the following linear program:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 8 \\ & 3x_1 + 4x_2 \leq 22 \\ & x_1 + 5x_2 \leq 23 \end{aligned}$$

Show that $(4/3, 10/3)$ is an optimal solution by providing a suitable feasible dual solution.

Solution

We have $c = (1,1)^T$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 4 \\ 1 & 5 \end{pmatrix}$, and $b = (6,8,22,23)^T$. The primal is

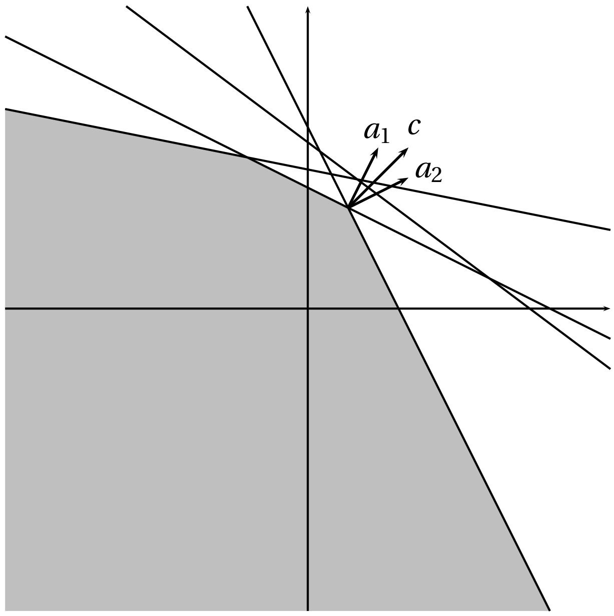
$$\begin{aligned} \max \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

and the corersponding dual is

$$\begin{aligned} \min \quad & b^T y \\ \text{subject to} \quad & A^T y = c \\ & y \geq 0 \end{aligned}$$

Note that every feasible solution to the dual gives a conic combination of the vector c using the columns of A.

Let a_1 and a_2 be the first two columns. The solution $(4/3, 10/3)$ satisfies their constraints with equality. If we draw the solution it looks like this:



Observe that c is in the cone of a_1 and a_2. Thus we know that the following system

$$\begin{aligned} 2y_1 + y_2 &= 1 \\ y_1 + 2y_2 &= 1 \end{aligned}$$

will have a nonnegative solution, namely $(\frac{1}{3}, \frac{1}{3})$. This gives the feasible solution $(\frac{1}{3}, \frac{1}{3}, 0, 0)^T$ to the dual LP.

Observe that $(4/3, 10/3) \cdot c = (\frac{1}{3}, \frac{1}{3}, 0, 0) \cdot b = \frac{14}{3}$.

Due to weak duality, our dual solution thus certifies that the primal solution is optimal.