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Introduction to Discrete Optimization

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Solutions 7

Exercise 1

Suppose that there are N available currencies, and assume that one unit of currency i can be exchanged for r_{ij} units of currency j. (Naturally we assume that $r_{ij} > 0$.) There are also certain regulations that impose a limit u_i on the total amount of currency i that can be exchanged on any given day (At most u_i units of currency i can be changed to any other currency).

Suppose that we start with B units of currency 1 and that we would like to maximize the number of units of currency N that we end up with at the end of the day, through a sequence of currency transactions. Provide a linear programming formulation of this problem. Assume that for any sequence $i_1, \dots, i_k$ of currencies, we have $r_{i_1 i_2} \cdot r_{i_2 i_3} \cdots r_{i_{k-1} i_k} \cdot r_{i_k i_1} \le 1$, which means that wealth cannot be multiplied by going through a cycle of currencies.

Solution

We introduce variables x_{ij} for each $i = 1, \dots, N$ and $j = 1, \dots, N$ which model the amount of currency i exchanged for currency j.

Since the amount of currency i that can be exchanged is limited by u_i, we get the constraints

$$\sum_{j=1}^{N} x_{ij} \le u_i, \;\; \forall i = 1, \dots, N.$$

For each currency $i \neq 1$, we cannot exchange more currency from i to other currencies than we changed to i before. This is modelled by the constraints:

$$\sum_{j=1}^{N} r_{ji} x_{ji} \ge \sum_{k=1}^{N} x_{jk}, \;\; \forall i = 2, \dots, N.$$

Since by assumption there are no profitable cycles, there is no reason to exchange currency from N to any other currency, or to change from any currency to 1. Thus we can set $x_{i1} = 0$ and $x_{Ni} = 0$ for each $i = 1, \dots, N$.

The amount of currency we can change from 1 is then upper bounded by the initial value of B:

$$\sum_{j=1}^{N} x_{1j} \le B.$$

Finally the amount of currency N is given by

$$\sum_{j=1}^{N} r_{jN} x_{jN}$$

which is our objective to maximize.

Summarizing, we get the following LP:

$$
\begin{array}{rll}
\max & \sum_{j=1}^{N} r_{jN} x_{jN} & \\
\text{subject to} & \sum_{j=1}^{N} x_{ij} \le u_i, & \forall i = 1,\dots,N \\
& \sum_{j=1}^{N} x_{1j} \le B & \\
& \sum_{j=1}^{N} r_{ji} x_{ji} \ge \sum_{k=1}^{N} x_{jk}, & \forall i = 2,\dots,N \\
& x_{i1} = 0,\ \ x_{Ni} = 0 & \forall i = 1,\dots,N \\
& x_{ij} \ge 0 & \forall i = 1,\dots,N,\ \ j = 1,\dots,N
\end{array}
$$

Exercise 2

Solve the following linear program using the two-phase simplex method:

$$
\begin{array}{rc}
\min & 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 \\
\text{subject to} & x_1 + x_2 + 4x_4 + x_5 = 2 \\
& x_1 + 2x_2 + -3x_4 + x_5 = 2 \\
& x_1 - 4x_2 + 3x_3 = 1 \\
& x_1, x_2, x_3, x_4, x_5 \ge 0
\end{array}
$$

During the first phase, let the following indices enter the basis in this order: $1, 2, 5$. Use the lexicographic pivoting rule to decide which index will leave the basis in each step.

For each iteration, give the basis and the simplex tableau.

Solution

In the first phase of the two-phase simplex algorithm our objective is to find an initial solution to the linear program. We introduce artificial variables s, t, and u and solve the linear program

$$
\begin{array}{rc}
\min & s + t + u \\
\text{subject to} & x_1 + x_2 + 4x_4 + x_5 + s = 2 \\
& x_1 + 2x_2 + -3x_4 + x_5 + t = 2 \\
& x_1 - 4x_2 + 3x_3 + u = 1 \\
& x_1, x_2, x_3, x_4, s, t, u \ge 0
\end{array}
$$

An initial solution for this LP is $x_1 = x_2 = x_3 = x_4 = x_5 = 0$, $s = t = 2$ and $u = 1$. The basic variables are s, t and u and the corresponding simplex tableau looks as follows:

x_1	x_2	x_3	x_4	x_5	s	t	u	
-3	1	-3	-1	-2	0	0	0	-5
1	1	0	4	1	1	0	0	2
1	2	0	-3	1	0	1	1	2
1	-4	3	0	0	0	0	0	1

As suggested in the exercise, we let x_1 enter the basis. Thus u has to leave and we get:

x_1	x_2	x_3	x_4	x_5	s	t	u	
0	-11	6	-1	-2	0	0	3	-2
0	5	-3	4	1	1	0	-1	1
0	6	-3	-3	1	0	1	-1	1
1	-4	3	0	0	0	0	0	1

Now, as suggested in the exercise, we let x_2 enter the basis. Thus t has to leave and we get:

x_1	x_2	x_3	x_4	x_5	s	t	u	
0	0	$\frac{1}{2}$	$-\frac{13}{2}$	$-\frac{1}{6}$	0	$\frac{13}{6}$	$\frac{7}{6}$	$-\frac{1}{6}$
0	0	$-\frac{1}{2}$	$\frac{13}{2}$	$\frac{1}{6}$	1	$-\frac{5}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
1	0	1	2	$\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{5}{3}$

Finally, as suggested in the exercise, we let x_5 enter the basis. According to the lexicographic pivoting rule s has to leave. We get:

x_1	x_2	x_3	x_4	x_5	s	t	u	
0	0	0	0	0	1	1	1	0
0	0	-3	39	1	6	-5	-1	1
0	1	0	-7	0	-1	1	0	0
1	0	3	-28	0	-4	4	1	1

Perturbing the columns we get:

x_1	x_2	x_3	x_4	x_5	s	t	u	
0	0	0	0	0	1	1	1	0
1	0	3	-28	0	-4	4	1	1
0	1	0	-7	0	-1	1	0	0
0	0	-3	39	1	6	-5	-1	1

All artificial variables have been driven out of the basis, thus we can remove them from the system again. We now have found the feasible solution $x = (1, 0, 0, 0, 1)$ for the original LP and the initial tableau coresponding to the basis $B = (1, 2, 5)$:

x_1	x_2	x_3	x_4	x_5	
0	0	*	*	0	*
1	0	3	-28	0	1
0	1	0	-7	0	0
0	0	-3	39	1	1

We still need to compute the reduced costs w.r.t. the original costs to start phase 2 of the simplex algorithm. We have

$$\bar{c}(3) = c(3) - c_B^T A_B^- 1 A_j = 3 - (2, 3, -2) \cdot \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} = -9$$

and

$$\bar{c}(4) = c(4) - c_B^T A_B^{-} 1 A_j = 3 - (2,3,-2) \cdot \begin{pmatrix} -28 \\ -7 \\ 39 \end{pmatrix} = 158.$$

Moreover $c^T x = 2 - 2 = 0$. Thus we get the following tableau:

x_1	x_2	x_3	x_4	x_5	
0	0	-9	158	0	0
1	0	3	-28	0	1
0	1	0	-7	0	0
0	0	-3	39	1	1

Only the reduced costs of x_3 are nonnegative, thus we let x_3 enter and x_1 has to leave. We get:

x_1	x_2	x_3	x_4	x_5	
3	0	0	74	0	3
$\frac{1}{3}$	0	1	$-\frac{28}{3}$	0	$\frac{1}{3}$
0	1	0	-7	0	0
1	0	0	11	1	2

All reduced costs are nonnegative. Thus we have found an optimal solution: $x = (0,0,\frac{1}{3},0,2)^T$ of objective value -3.

Exercise 3
Solve the following simplex tableau using the dual simplex method.

x_1	x_2	x_3	x_4	x_5	
3	2	1	0	0	0
-2	2	-1	1	0	-3
-2	-1	1	0	1	-1

For each iteration, give the basis and the simplex tableau.

Solution
The initial basis is $B_0 = \{4,5\}$. Both x_4 and x_5 are negative and can be chosen to leave the basis. We choose 4. The ratio $\frac{1}{1}$ is smaller than $\frac{3}{2}$, and thus 3 enters the basis. We get:

x_1	x_2	x_3	x_4	x_5	
1	4	0	1	0	-3
2	-2	1	-1	0	3
-4	1	0	1	1	-4

The new basis is $B_1 = \{3,5\}$. Now 5 has to leave the basis and only 1 can enter. We get:

x_1	x_2	x_3	x_4	x_5	
0	$\frac{17}{4}$	0	$\frac{5}{4}$	$\frac{1}{4}$	-4
0	$-\frac{3}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	1
1	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{1}{4}$	1

The new basis is $B_2 = \{1,3\}$. The associated basic solution is feasible and the reduced costs are nonnegative. Thus the solution $(1,0,1,0,0)^T$ is optimal.

Exercise 4
Consider the following simplex tableau:

x_1	x_2	x_3	x_4	x_5	
δ	-2	0	0	0	-10
-1	η	1	0	0	4
α	-4	0	1	0	1
γ	3	0	0	1	β

The current basic variables are x_3, x_4, x_5. The entries $\alpha, \beta, \gamma, \delta, \eta$ in the tableau are unknown parameters.

For each one of the following statements, find some parameter values that will make the statement true.

1. The current solution is feasible but not optimal.
2. The current solution is optimal.
3. The optimal cost is $-\infty$.

Solution

1. If $\beta \geq 0$, then the solution is feasible. If $\beta > 0$, then we will improve the solution by bringing x_2 into the basis no matter which values the other parameters have.
2. We need $\beta = 0$ since otherwise the solution is not optimal as seen in 1. If we further set $\eta \leq 0$, and $\delta, \alpha, \gamma \geq 0$, then the next iteration of the simplex algorithm will bring x_2 into the basis and drop x_5 out of it. The objective value will not change and all reduced costs will be nonnegative afterwards. Thus the current solution is optimal.
3. We set $\delta < 0$ and $\alpha, \gamma \leq 0$. If we attempt to bring x_1 into the basis, this shows that the linear program is unbounded.

Exercise 5
Consider the following linear programming problem with a single constraint:

$$\begin{array}{rl} \min & \sum_{i=1}^n c_i x_i \\ \text{subject to} & \sum_{i=1}^n a_i x_i = b \\ & x_i \geq 0, \qquad i = 1, \ldots, n. \end{array}$$

Here $b, a_i, c_i \in \mathbb{R}$ for each $i = 1, \ldots, n$

1. Derive a simple test for checking the feasibility of this problem.
2. Assuming that the optimal cost is finite, develop a simple method for obtaining an optimal solution directly.

Solution

1. We distinguish three cases depending on the value of b.
   - If $b = 0$, then the system is always feasible since $x = 0$ is a feasible solution.
   - If $b < 0$: Let $a_i < 0$ for some i. Then $x_i = \frac{b}{a_i}$ and $x_j = 0$ for all $j \neq i$ is a feasible solution. If $a_i \geq 0$ for all $i = 1, \dots, n$, then the system is infeasible since $\sum_{i=1}^n a_i x_i \geq 0$ for each $x \geq 0$.
   - If $b > 0$, the system is feasible if and only if there is an i with $a_i > 0$. The proof is analogous to the case $b < 0$.

2. We first consider the case that $b = 0$. Assume that there is a solution x such that $\sum_{i=1}^n c_i x_i < 0$. Then for every $k \in \mathbb{N}$, the vector $k \cdot x$ is a solution of value $k \cdot \sum_{i=1}^n c_i x_i$ which shows that the optimal cost is not finite in contradiction to the assumption.

   Therefore there is no solution with negative objective value, and thus $x = 0$ is an optimal solution.

   Now let $b > 0$. Define $M := \{i = 1, \dots, n \ : \ a_i > 0\}$. Let i^* be the index such that the minimum $\min_{i \in B}\{\frac{b}{a_i} \cdot c_i\}$ is attained. We want to prove that x^* with $x_{i^*} = \frac{b}{a_{i^*}}$ and $x_i = 0$ for $i \neq i^*$ is an optimal solution.

   Let x be an optimal solution to the system such that $x_i = 0$ for each $i \notin M$. Such an optimal solution must exist because we assumed that the optimal cost is finite.

   For each $i \in M$, let $\lambda_i := \frac{a_i}{b} x_i$, thus $x_i = \frac{b}{a_i} \lambda_i$.

   Since x is a solution to the system we have

   $$b = \sum_{i \in B} a_i x_i = \sum_{i \in B} a_i \frac{b}{a_i} \lambda_i = b \cdot \left(\sum_{i \in B} \lambda_i \right).$$

   Thus $\sum_{i \in B} \lambda_i = 1$.

   The costs of x are given as

   $$\sum_{i \in B} c_i x_i = \sum_{i \in B} \frac{b}{a_i} c_i \lambda_i \geq \sum_{i \in B} \frac{b}{a_{i^*}} c_{i^*} \lambda_i = \frac{b}{a_{i^*}} c_{i^*} \cdot \sum_{i \in B} \lambda_i = \frac{b}{a_{i^*}} c_{i^*}.$$

   This shows that x^* is optimal.

   For $b < 0$ one can analogously prove that x^* with $x_{i^*} = \frac{b}{a_{i^*}}$ and $x_i = 0$ for $i \neq i^*$, where i^* is the index such that the minimum $\min_{i : a_i < 0}\{\frac{b}{a_i} \cdot c_i\}$ is attained is an optimal solution.