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Introduction to Discrete Optimization

Spring 2009

Solutions 11

Exercise 1

Let $D = (V, A)$ be a directed graph with capacities $u\colon A \to \mathbb{Q}_{\geq 0}$, costs $c\colon A \to \mathbb{Q}$ and external flow $b\colon V \to \mathbb{Q}$.

1. Show how to transform the minimal cost network flow problem for D, u c and b to a minimal cost network flow problem on a graph D' with functions u', c' and b' such that D' does not contain a pair of reverse arcs.

   Explain how to transform an optimal solution for the MCNFP on D' to an optimal solution for the MCNFP on D.

2. Show that there is no feasible flow in D with capacities u and external flow b unless $\sum_{v \in V} b(v) = 0$ holds.

3. Why can we assume that the network has a path from i to j for all $i \neq j \in V$ which is incapacitated?

Solution

1. The transformation works as follows: We create two copies V_1 and V_2 of V. For each node $v \in V$, we connect its incoming arcs to its copy in V_1, and each outgoing arc to V_2. We further introduce new edges pointing from the copy of v in V_1 to the copy in V_2 of zero costs and infinite capacity. The external flow of the nodes in V_1 is copied from V, and the external flow of the nodes of V_2 is zero.

   Formally, we define two sets V_1 and V_2 and bijections $\varphi_1 : V \to V_1$ and $\varphi_2 : V \to V_2$ and set $V' = V_1 \dot\cup V_2$. We define $A^* := \{(\varphi_2(v), \varphi_1(w)) \,:\, (vw) \in A\}$ and set

   $$A' := A^* \cup \{(\varphi_1(v), \varphi_2(v)) \,:\, v \in V\}$$

   to obtain the network $D' = (V', A')$. The capacities $u' : A' \to \mathbb{Q}_{\geq 0}$ are defined as

   $$u'(v, w) = \begin{cases} u\left(\varphi_2^{-1}(v), \varphi_1^{-1}(w)\right), & \text{if } v \in V_2 \text{ and } w \in V_1 \\ \infty, & \text{else.} \end{cases}$$

   The costs $c' : A' \to \mathbb{Q}$ are defined as

   $$u'(v, w) = \begin{cases} c\left(\varphi_2^{-1}(v), \varphi_1^{-1}(w)\right), & \text{if } v \in V_2 \text{ and } w \in V_1 \\ 0, & \text{else.} \end{cases}$$

Finally the external flow $b': V' \to \mathbb{Q}$ is defined as

$$b'(v) = \begin{cases} b\left(\varphi_1^{-1}(v)\right), & \text{if } v \in V_1 \\ 0, & \text{else.} \end{cases}$$

Note that there is a 1 to 1 correspondence between the arcs A and the arcs in A^*. Given a feasible flow f in D, we obtain a feasible flow f' in D' of the same costs by assigning the same flow to the corresponding arcs in A^*. For the other arcs, i.e. the arcs of type $(\varphi_1(v), \varphi_2(v))$ we set the flow to $f(\delta^{in}(v)) - b(v)$.

Given a feasible flow f' in D', we obtain a feasible flow f in D of the same costs by assigning the flow on the arcs of A^* to the corresponding arcs in A.

2. Let f be a feasible flow. Thus we have $b(v) = f(\delta^{out}(v)) - f(\delta^{in}(v))$ for each $v \in V$. This gives

$$\sum_{v \in V} b(v) = \sum_{v \in V} \left(\delta^{out}(v) - \delta^{in}(v)\right) = -\sum_{v \in V} excess_f(v) = -excess_f(V) = 0.$$

3. Let $M := \sum_{a \in A, c(a)>0} u(a) \cdot c(a) + 1$. Observe that if the MCNFP on D has a feasible solution, its value is less than M. Thus, if we add arcs (i, j) of infinite capacity for each pair $i, j \in V$ and set its costs to M, no such arc will be used in the optimal solution of the modified MCNFP unless the original MCNFP had no feasible solution.

Exercise 2

Consider a hockey tournament with n teams. The playing schedule is given as a list with m entries, each of the form (u, v, k), meaning that team u and team v play k times against each other. No match can end in a tie, i.e. each game has a winner and a looser. Let x_i be the number of wins of team i.

Given a vector $(x_1, \dots, x_n)$, we want to know if it reflects a possible tournament outcome. Show how to model this as a network flow problem. Introduce a digraph D, capacities u and an external flow b such that $(x_1, \dots, x_n)$ is a possible tournament outcome if and only if there is a feasible flow in D subject to u and b.

Solution

We introduce a node for each team and for each entry on the playing schedule. For each entry $i:\ u, v, k$ we add two arcs (u, i) and (v, i), each of capacity k. The external flow for node i will be set to $-k$, and the external flow for each team u is x_u, the number of times the team has won.

If $(x_1, \dots, x_n)$ is a possible tournament outcome, then we obtain a feasible flow as follows: For each entry $i : u, v, k$ on the playing schedule, we set $f(u, i)$ to the number of times u has won against v, and $f(v, i)$ to the number of times v has won against u.

Conversely, if we have a feasible flow, then it directly yields a possible tournament outcome: For each entry $i : u, v, k$ on the playing schedule, team u has won $f(u, i)$ times against team v and lost $f(v, i)$ times.

Exercise 3
Let $D = (V, A)$ be a directed graph with capacities $u: A \to \mathbb{Q}_{\geq 0}$ and external flow $b: V \to \mathbb{Q}$.

Explain how to find a feasible flow in D subject to u an b efficiently or assert that no feasible flow exists.

Hint: Use a maximum $s-t$-flow algorithm on an auxiliary network $D' = (V', A')$.

Solution
Let $V^+ := \{v \in V : b(v) > 0\}$ and $V^- := \{v \in V : b(v) < 0\}$. Set $V' := V \cup \{s, t\}$ and $A' := A \cup A^*$, where $A^* := \{(s, v) : v \in V^+\} \cup \{(v, t) : v \in V^-\}$ and introduce capacities

$$u': A' \to \mathbb{Q}_{\geq 0},\ u'(a) = \begin{cases} u(a), & \text{if } a \in A \\ b(v), & \text{if } a = (s, v) \\ -b(v), & \text{if } a = (v, t). \end{cases}$$

Set $D' = (V', A')$. We claim that there is an $s - t$ flow of value $b(V^+)$ in D' with capacities u' if and only if there is a feasible flow in D subject to u and b.

First assume that there is an $s - t$-flow f' of value $b(V^+)$. By construction, each arc (s, v) is then fully saturated, i.e. $f'(s, v) = b(v)$ for each $(s, v) \in A'$. The same is true for each arc (v, t), i.e. $f'(v, t) = -b(v)$ for each $(v, t) \in A$.

Since f' is a feasible flow, the flow conservation constraint

$$f'(\delta^{out}_{D'}(v) - f'(\delta^{in}_{D'}(v)) = 0$$

is satisfied for each $v \in V$. Thus if we define $f : A \to \mathbb{Q}_{\geq 0}$, $f(a) = f'(a)$, then for each $v \in V^+$ we get $f(\delta^{out}_D(v) - f(\delta^{in}_D(v)) - b(v) = 0$ and $f(\delta^{out}_D(v) - b(v) - f(\delta^{in}_D(v)) = 0$ for each $v \in V^-$. This shows that f is a feasible flow in D subject to u and b.

Now assume that we have a feasible flow f in D subject to u and b. We define

$$f': V' \to \mathbb{Q}_{\geq 0},\ f'(a) = \begin{cases} f(a), & \text{if } a \in A \\ b(v), & \text{if } a = (s, v) \\ -b(v), & \text{if } a = (v, t) \end{cases}$$

Similarly to the argumentation above one gets that f' is a feasible flow in D' of value $b(V^+)$.

Exercise 4
A *matching* in an undirected graph $G = (V, E)$ is a subset $M \subseteq E$ of the edges such that no two edges in M share a common node of V.

Given a cost function $c: E \to \mathbb{R}$, the *matching-problem* is to find a matching that maximizes $c(M) = \sum_{e \in M} c(e)$.

A graph is *bipartite*, if there is a partition V_1, V_2 of V, i.e. we have $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$, such that there are no edges between nodes of V_1 and no edges between nodes of V_2.

Show how to formulate the matching-problem for bipartite graphs as a *min-cost-circulation* problem.

Solution
Let $G = (V, E)$ be a bipartite graph with V_1 and V_2 as above. Define $D' = (V', A')$, where $V' :=$

$V \cup \{s,t\}$ and the arcs are given as $A' := \{(v_1, v_2) : v_1 \in V_1, v_2 \in V_2, \{v_1, v_2\} \in E\} \cup \{(s,v) : v \in V_1\} \cup \{(v,t) : v \in V_2\} \cup \{(t,s)\}$. Each arc gets a capacity of 1, except for the arc (t,s) which has infinite capacity. The arcs of the form (v_1, v_2) get costs $-c(\{v_1, v_2\}$, every other arc gets costs 0.

Every matching in G of weight w corresponds to a feasible flow in D' of costs $-w$ and vice versa.